UNIVERSITY OF CALIFORNIA, LOS ANGELES

Portfolio Analysis Based On A Simplified Model Of The Relationships Among Securities

A dissertation submitted in partial satisfaction of the requirements for the degree doctor of Philosophy in Economics

by

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

Figure		Page

I. INTRODUCTION

A. An Overview of the Study

This dissertation reports the results of an investigation of the process of portfolio selection under certain simplified assumptions about the relationships among securities. It uses a set of mathematical equations called the diagonal model (for a reason which will be made clear later) to represent these relationships. The simplicity of the diagonal model makes it possible to develop correspondingly simple theories concerning the process of portfolio selection. This study develops and tests such theories in both normative and positive applications.

The key issue in the normative application of a theory of portfolio selection concerns the manner in which predictions about the future are obtained. We will distinguish two general techniques with which predictions can be made: objective techniques, in which predictions are based on specific data and follow a set of rules which can be completely specified; and subjective techniques, in which some or all of the prediction process cannot be so specified. This study examines both techniques; Chapter III describes a test of portfolio analysis based on objective techniques, while Chapter IV concerns an experiment involving subjective prediction techniques.

The positive applications of the theoretical technique developed in the study are described in Chapter V, in which a theory of market equilibrium is derived from the assumption that investors feel that the important relationships among securities are those specified by the diagonal model. The study is then briefly summarized in Chapter VI.

The diagonal model itself is described in Chapter II, as is the formulation of the portfolio-selection problem associated with it. The technique of portfolio selection used in this study is that developed by Markowitz.[1] Since the Markowitz technique forms a major element of the theoretical apparatus used in the study, it seems desirable to devote considerable attention to it at the outset. The remainder of this chapter is devoted to this task.

B. The Markowitz Approach to Portfolio Selection

Economic theory postulates that an individual possesses resources which he can combine to produce any of a number of alternative bundles of goods. Of all such possible combinations of goods, there is a set which is efficient. An efficient combination is one which has the most of one good attainable with given amounts of all other goods. Put another way, a combination is efficient if no other attainable combination contains more of at least one good and no less of any good.

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[1] Harry M. Markowitz, Portfolio Selection, Efficient Diversification of Investments (New York: John Wiley and Sons, Inc., 1959). For an earlier version see his "Portfolio Selection," The Journal of Finance, XII (March, 1952), 77-91. A simple exposition of the technique is given in John Frederick Weston and William Beranek, "Programming Investment Portfolio Construction," The Analysts' Journal, XI (May, 1955), 51-55.

Economic theory also assumes that the individual correctly determines those efficient combinations which he can obtain. Having found the possibilities available to him, he chooses the one which maximizes his utility.

Markowitz approaches the subject of portfolio selection in an analogous fashion. The investor has a given amount of wealth with which he can purchase various combinations of investment media. The future wealth of the investor is directly related to the particular combination of investment media in which he places his present wealth. Markowitz assumes that the investor views the future in probabilistic terms. Thus, any of the attainable combinations has an associated probability distribution indicating the investor's beliefs concerning his future wealth if the combination in question is selected.

The terms "yield" and "return" will be used throughout this dissertation to refer to the ratio of future wealth to present wealth. Thus the yield of a security over one year is found by dividing (a) the price of the security at the end of the year plus all dividends paid during the year by (b) the price of the security at the beginning of the year.

For the analysis of portfolios, it is desirable to relate probability distributions of future wealth to the present wealth required to obtain them; probability distributions of yield meet this requirement. Markowitz suggests that such distributions be described with two parameters:[2] the expected yield (E), and the variance of yield (V).[3] Expected yield is a good: larger values give the investor greater utility. On the other hand, large amounts of variance are undesirable: smaller values give the investor greater utility.

According to Markowitz, the investor should choose from among his attainable combinations those which are efficient with respect to these two parameters. A combination is efficient if no other has either (1) the same E and a lower V or (2) the same V and a greater E. Having selected the set of efficient combinations, the investor chooses the one among them which maximizes his utility.

Probability beliefs about a combination of investment media can be built up from corresponding beliefs about individual investments. For ease of exposition, the term "security" will be used for any investment -- e.g., a stock, a bond, cash, real estate, etc. The investor is assumed to form probability beliefs about each security available to him. These beliefs include not only the

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[2]A parameter is an algebraic symbol used to indicate a quantity the value of which will differ among applications.

[3]Actually, Markowitz prefers the semi-variance (the average of the squared deviations below the expected value) to variance. However, the two measures give similar results under certain common circumstances. It can be shown that if the diagonal model portrays the true relationships among securities, variance will be an adequate measure if the probability distribution of the future level of the security market is normally distributed. Since no computing algorithm now exists for portfolio analysis using the semi-variance, the study deals only with variance as a measure of risk.

mean and variance of the yield of the security itself, but also its relationship with each of the other securities. An investor could specify the expected yield of a security, the variance of that yield, and the covariance between its yield and that of each of the other securities. Given these estimates, the characteristics of the associated probability beliefs about any combination of securities-- or portfolio -- can be determined. Let E_i be the expected yield of the i^{th} security and C_{ij} the covariance between its yield and that of the j^{th} security (C_{ii} represents the variance of i). Then, if X_i is the proportion of the value of a portfolio invested in the i^{th} security, the mean and variance of the portfolio's yield are:

$$E = \sum_i X_i E_i$$

and

$$V = \sum_i \sum_j X_i X_j C_{ij}.$$

An example may help to clarify the meanings of the parameters. Assume there are two securities, the first of which is expected to yield 10% per annum while the second is expected to yield 15%. Then:

$$E_1 = 1.10$$

$$E_2 = 1.15$$

Assume further that the variance of the first security is 0.09. This implies a standard deviation of 0.3, which, in turn, implies that there is a probability of only 0.10 that the actual yield in any given year will fall below 0.715 ($E - 1.282\sigma$). Thus there is a

10% chance that the purchase of this security would lead to a loss of more than 28% of the initial value. Let the variance of the second security be 0.16 and the covariance between them 0.05:

$$C_{11} = 0.09$$

$$C_{22} = 0.16$$

$$C_{12} = C_{21} = 0.05$$

If a portfolio of $100 were invested in the two securities so that $30 of the value were held in Security 1 with the remaining $70 in Security 2, we would have:

$$X_1 = 0.3$$

$$X_2 = 0.7$$

The expected yield of the portfolio would be:

$$E = 0.3 \cdot 1.10 + 0.7 \cdot 1.15 = 1.135$$

and its variance would be:

$$\begin{aligned} V &= 0.3 \cdot 0.3 \cdot 0.09 + 0.3 \cdot 0.7 \cdot 0.05 \\ &\quad + 0.7 \cdot 0.3 \cdot 0.05 + 0.7 \cdot 0.7 \cdot 0.16 \\ &= .1075 \end{aligned}$$

Markowitz separates the process of investment choice into three stages. The first, security analysis, is the process by which the investor obtains predictions about the future performance of securities: the expected yields, variances, and covariances described above. The second, portfolio analysis, is the process by which the properties of all attainable combinations of securities are ascertained, and the set of efficient combinations selected. In the third stage, portfolio selection, the investor chooses the efficient portfolio which maximizes his utility.

Markowitz's major contribution to the process of investment selection deals with the second of these three stages. Selection of the set of efficient portfolios can be viewed as a series of problems of the form: minimize V for a given E. A portfolio is specified by the amounts of each security included in it -- in terms of our previous notation, a vector $[X] = (X_1, X_2, \ldots, X_n)$ where X_i is the amount of the portfolio invested in the i^{th} security. Since the total amount invested must equal one,[4] we require:

$$\sum_{i=1}^{n} X_i = 1$$

As shown above, E is a linear function of [X], while V is a quadratic function of [X]. Thus the selection of a portfolio with minimum V for a given E requires a process which will find a value of [X] which minimizes a quadratic function subject to two linear constraints.[5]

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[4] This applies only in the "standard" case where no borrowing is permissible. We shall use this assumption throughout the initial chapters since it makes computation somewhat simpler. The possibility of borrowing and lending is easily taken into account after the portfolio analysis is completed, as we will indicate in Chapter V. The only activity denied by this formulation is negative purchase of a particular security -- "short selling."

[5] $\sum_{i=1}^{N} X_i = 1$ and $E = \bar{E}$.

The process of minimizing a quadratic function subject to linear constraints is known as quadratic programming,[6] for which several solution techniques are now available. By repeating the process for various values of E, all efficient portfolios can be determined.[7]

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[6]No algorithm has been developed for the associated problem which maximizes a linear function subject to linear and quadratic constraints.

[7]In practice, the problem is solved in a slightly different manner. See Appendix A.

II. THE DIAGONAL SECURITY MODEL

A. Important Attributes of Securities

Discussions with security analysts indicate that three aspects of the future performance of a security are usually considered explicitly. First, a security is classified as either high-yield or low-yield on the basis of the analyst's "best guess" concerning its future performance. Such an estimate corresponds to the expected-yield quantity (E) required for the portfolio-analysis problem.

The second aspect which receives attention in most security analyses is the risk that the most likely return may not be realized. Securities are considered more speculative, the greater is this risk. The variance parameter required by portfolio analysis makes explicit this notion of risk.

The third aspect of the performance of a security which is often considered explicitly is its relationship to the security market (and/or the economy) in general. Securities which rise and fall with the market are considered more sensitive or cyclical than those which are little affected by such broad movements. Thus the risk of a security is often considered to be due to two factors: the risk associated with the firm itself, and the risk of a market (or general business) decline with an associated effect on the firm in question.

In the diagonal model these three attributes -- expected yield, risk, and dependence on the market -- are made explicit. The simplicity of the model derives from the fact that these attributes are the only ones taken into account. Evidence presented in the next chapter suggests that any loss in precision due to this simplicity is likely to be small. It is shown below that the cost of portfolio analysis can be greatly reduced when the diagonal model is used. For these reasons the model appears worthy of the detailed examination it receives in this dissertation.

B. The Diagonal Model[1]

The basic equation of the diagonal model relates the yield of a security to its own attributes and to the performance of some index of market activity:

$$Y_i = A_i + B_i \cdot I + w_i \tag{1}$$

where A_i and B_i are parameters, and w_i is a random variable with an expected value of zero and a variance of Q_i. The parameter (I) represents the level of an index of some activity considered to be of major importance in determining the yields of most securities. In this study we use the level of the security market for this index. A number of alternative attributes are of interest and may be incorporated into future work.[2] Such measures can be substituted for the market level without altering the formulation of the

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[1]This model is one of a number suggested by Markowitz; see his Portfolio Selection, pp. 96-101.

[2]A particularly interesting candidate is the general price

diagonal model presented here; on the other hand, if they are to be incorporated in addition to the market index, the model will have to be expanded. Models which relate the yield of a security to more than one such factor will undoubtedly prove valuable; however, in this study we will restrict our attention to the simple model in which but one element of this type influences the yields of most securities.

For a number of problems it is convenient to measure (I) in terms of deviations (either relative or absolute) from its expected value. However, we will formulate the model for the general case in which the expected value of (I) is non-zero, so that any desired measurement can be used.

Equation (1) constitutes the basic assumption of the diagonal model. Security analysis, in this model, involves the specification of three parameters for each security: A_i, B_i, and Q_i. These

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level of the economy. The wealth of a firm can be shown to be affected by the price level and the relative importance of real and monetary assets and liabilities in its financial structure. In particular, if (M) is the firm's net monetary creditor position (monetary assets less monetary liabilities) and (R) its net real asset position (real assets less real liabilities), then the effect of changes in the price level on the firm's wealth can be shown to be proportional to R/(R+M). The diagonal model can be utilized to reflect this relationship by using R/(R+M) as an estimate of B_i for each security, with (I) representing an index of the general price level in the economy. For the general model of the effects of inflation on the wealth position of a firm, see R. Kessel, "Inflation-caused Wealth Redistribution," American Economic Review, XLVI (March, 1956), 128-141.

parameters specify the expected return, responsiveness to the market, and risk attributes discussed in the previous section. The only additional analysis required concerns the probable future course of the market. The analyst must specify an expected value and a variance for (I). Let:

A_{n+1} = the expected value of I ($=\bar{I}$)

Q_{n+1} = the variance of I ($=V_I$)

The reason for this notation will become clear later in the chapter.

The diagonal model assumes that securities are unrelated except through their common dependence on the market; further, the random elements (w_i) are not related to the level of the index (I). Thus:

$$\text{Covariance } (w_i, w_j) = 0 \quad \text{for all i and j}$$

$$\text{Covariance } (w_i, I) = 0 \quad \text{for all i.}$$

These assumptions complete the information necessary for obtaining the elements required by the Markowitz portfolio-analysis technique from the parameters of the diagonal model.

The expected return of a security (E_i) and its variance (V_i) are simply:

(2) $$E_i = A_i + B_i \cdot A_{n+1}$$

and

(3) $$V_i = (B_i)^2 \cdot Q_{n+1} + Q_i$$

The diagonal model does not explicitly specify covariances among securities, but they are implied by common relationships with (I). If a fall in the market would reduce the returns on two securities, ceteris paribus, then their returns are correlated. Let C_{ij} be the covariance between Securities i and j. Then:

(4) $$C_{ij} = B_i \cdot B_j \cdot Q_{n+1}$$

The derivation of this relationship is shown in the footnote.[3]

To summarize, the diagonal model requires specifications of three parameters for each security, and two parameters for the level of the market in general. From these, the required inputs for the portfolio-analysis problem can be derived. If there are N securities, the N expected values can be computed according to Eq. (2), the N variance terms computed using Eq. (3), and the $(N^2-N)/2$ covariance terms calculated according to Eq. (4). Once these terms are derived, the portfolio analysis can proceed.

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[3] If z is the divergence of (I) from its expected value, Y_i would be expected to diverge from its expected value by an amount $B_i z$. Similarly, Y_j would be expected to diverge from its expected value by an amount $B_j z$. The cross-product of the two divergences is $(B_i z \cdot B_j z)$ or $(B_i B_j z^2)$. The covariance between i and j is the sum of all such cross-products, divided by the number of occurrences (N). Thus:

$$C_{ij} = \frac{1}{N} \sum_z B_i B_j z^2$$
$$= B_i B_j \frac{1}{N} \sum_z z^2$$

But: $$\frac{1}{N} \sum_z z^2 = V_I = Q_{n+1}$$

Therefore: $$C_{ij} = B_i \cdot B_j \cdot Q_{n+1}$$

C. The Associated Portfolio-Analysis Problem

Portfolio analysis involves repeated minimization of the variance of a portfolio for various expected returns. In the general case, the variance of a portfolio of N securities is given by:

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j C_{ij} ,$$

while its mean is:

$$E = \sum_{i=1}^{N} E_i$$

We have shown above that all the terms required for the general portfolio-analysis can be derived from the parameters of the diagonal model. However, considerable effort is required to derive $(N^2-N)/2$ covariances and then to prepare and perform the associated quadratic programming problem. In this section we will develop an alternative approach which greatly reduces the effort involved in performing portfolio analysis based on the diagonal model.

An investor devoting his funds to a security can be considered to be purchasing two separate investments. First, he acquires the unique characteristics of the security in question. Second, he acquires a portion of the security market in general. This is the case if the return on the security is at all connected with the performance of the market. This relationship can be seen if we rewrite Eq. (1). The basic formula of the model then becomes:

(5) $$Y_i = (A_i + w_i) + B_i \cdot I$$

The yield of a security is determined by three factors. A_i and w_i are unique to the security and are in no way related to the market. The third factor is not unique to the security; it is composed of two elements: B_i, the responsiveness of Y_i to the market level; and (I), the market level itself. Thus we can think of the investor who purchases a dollar's worth of Security i as having invested one dollar in $(A_i + w_i)$ and one dollar in $(B_i \cdot I)$.

Assume that an investor purchases two securities, i and j, dividing the total value of his portfolio between them in the proportions X_i and X_j, where $(X_i + X_j) = 1$. The yield on such a portfolio would be merely the weighted average of the yields of the two securities, with the relative amounts of the securities used as weights:

$$Y_p = X_i Y_i + X_j Y_j$$

Substituting the basic equation of the diagonal model, we have:

$$\begin{aligned} Y_p = {} & X_i(A_i + B_i \cdot I + w_i) \\ & + X_j(A_j + B_j \cdot I + w_j) \end{aligned}$$

Rearranging terms:

$$\begin{aligned} Y_p = {} & X_i(A_i + w_i) \\ & + X_j(A_j + w_j) \\ & + (X_i B_i + X_j B_j) I \end{aligned}$$

This formulation shows that the purchase of two securities can be viewed as an investment in three different media: the unique characteristics of Security i, the unique characteristics of Security j, and the influence of the market in general. The influence of the market on the yield of the portfolio partly depends on the influence which the market exerts on the securities in the portfolio. The effect of the market on the yield of Security i is shown by B_i; since X_i of the portfolio is invested in that security, the total influence of the market on the portfolio due to the holdings of Security i is given by (X_iB_i). Holdings of Security j give rise to an additional market influence of X_jB_j on the yield of the portfolio.

In the case of two securities, the total "investment" in the market is $(X_iB_i + X_jB_j)$. Let this total investment in the market be represented by X_{n+1}. In the general case we define this parameter as follows:

$$X_{n+1} = \sum_{i=1}^{N} X_iB_i \tag{6}$$

(X_{n+1}) may be viewed as a weighted-average responsiveness to the market level. (B_i) is the responsiveness of Security i to the market. The responsiveness of a portfolio is merely the weighted-average responsiveness of the individual securities, where the appropriate weights are the relative amounts invested. Note that this investment in the market results from investment in securities; it is not possible to devote a portion of a portfolio directly to

X_{n+1}. The portfolio is divided among the N securities $(\sum_{i=1}^{N} X_i = 1)$; such a division implies a particular value for X_{n+1}.

We have separated the influence of the market level on securities from their unique characteristics. We now need to revise our notation to account for the remaining effects on the yield of a security.

Let i* be the "basic security" whose yield is given by:

$$Y_{i*} = A_i + w_i$$

The mean and variance of such a security are given by:

$$E_{i*} = A_i$$

$$V_{i*} = Q_i$$

By assumption, the only relationships among securities arise from their common dependence on the market. "Basic securities" of the type defined above do not depend on the market; thus their covariances are zero. For all such securities:

$$C_{i*j*} = 0$$

Since we have assumed no correlation between the w parameters and (I), the covariance between each basic security and the $(n+1)^{st}$ security -- the market index -- is also zero:

$$C_{i*,n+1} = 0$$

We are now able to restate the portfolio-analysis problem in a manner which leads to a considerable reduction in effort. We have shown that an allocation of X_i of a portfolio to Security i can be considered an investment of X_i in the Basic Security i* and an additional investment of $(X_i B_i)$ in the market. For consistency in notation, let X_{i*} represent the investment in the Basic Security i*. Then:

$$X_{i*} = X_i \qquad \text{for } i \leq N$$

Recall that the sum of such investments represents the full portfolio, thus:

$$\sum_{i=1}^{N} X_{i*} = 1$$

The devotion of X_i of a portfolio to Security i brings about two effects. The first, an investment of X_{i*} in the Basic Security i*, has been described; the second, an investment of $(X_i B_i)$ in the market, is included in X_{n+1}, the term used to summarize the entire effect of the market on the portfolio. Again, for consistency in notation, we define:

$$X_{(n+1)*} = \sum_{i=1}^{N} X_i B_i \quad .$$

We have used A_{n+1} to represent the expected value of (I); the advantage of this notation can be seen in the following simple formula for the expected yield of a portfolio:

(7) $$E = \sum_{i=1}^{N+1} X_i A_i$$

The expected yield of a portfolio is simply the expected return of each basic security weighted with its importance in the portfolio, plus the expected market level times the parameter which relates the yield of the portfolio to that level. Proof of the validity of this formulation is given in the footnote.[4]

The variance of a portfolio can be expressed in a similar manner. Variance arises from investments in basic securities and from the effects of the market in general. Since none of these investment media are correlated with each other, the variance-covariance matrix for N basic securities and one $(N+1)^{st}$ security -- the market level -- contains non-zero entries only along the diagonal. It is this property which gives rise to the name used to

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[4] $E_i = A_i + B_i \cdot A_{n+1}$

$$E = \sum_{i=1}^{N} X_i E_i$$

Therefore: $$E = \sum_{i=1}^{N} X_i (A_i + B_i \cdot A_{n+1})$$

$$= \sum_{i=1}^{N} X_i A_i + A_{n+1} \sum_{i=1}^{N} X_i B_i$$

But: $$X_{n+1} \equiv \sum_{i=1}^{N} X_i B_i$$

Therefore: $$E = \sum_{i=1}^{N} X_i A_i + X_{n+1} A_{n+1} = \sum_{i=1}^{N+1} X_i A_i$$

describe the model. The general formula for the variance of a portfolio is given by:

$$(8) \qquad V = \sum_{i=1*}^{(N+1)*} \sum_{j=1*}^{(N+1)*} X_i X_j C_{ij}$$

But: $$C_{ij} = 0 \qquad \text{if } i \neq j$$

And, since we have defined (Q_{n+1}) as the variance of I:

$$C_{ij} = Q_i \qquad \text{if } i = j$$

Thus the formula for the variance of a portfolio can be written:

$$V = \sum_{i=1}^{N+1} (X_i)^2 \cdot Q_i$$

Proof of the validity of this formulation is shown in the footnote.[5]

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[5] $$C_{ii} = V_i = Q_i + (B_i)^2 Q_{n+1}$$

$$C_{ij} = B_i B_j Q_{n+1} \qquad \text{if } i \neq j$$

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i X_j C_{ij})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i X_j B_i B_j Q_{n+1}) + \sum_{i=1}^{n} (X_i^2 Q_i)$$

$$= Q_{n+1} \left[\sum_{i=1}^{n} (X_i B_i) \sum_{j=1}^{n} (X_j B_j) \right] + \sum_{i=1}^{n} (X_i^2 Q_i)$$

But: $$\sum_{i=1}^{n} (X_i B_i) = \sum_{i=1}^{n} (X_j B_j) \equiv X_{n+1}$$

Therefore: $$V = Q_{n+1}(X_{n+1})^2 + \sum_{i=1}^{n} X_i^2 Q_i = \sum_{i=1}^{n+1} X_i^2 Q_i$$

This completes our re-formulation of the portfolio-analysis problem associated with the diagonal model. By defining a new variable -- X_{n+1} -- according to Eq. (6), it is possible to represent the mean and variance of a portfolio with relatively simple functions of the original parameters of the model. The advantages of these are indicated in the next section.

D. The Diagonal Model Portfolio-Analysis Code

A number of computer codes are now available for the solution of quadratic programming problems. One such program, the RAND Quadratic Programming Code,[6] was used to solve some of the portfolio analysis problems in this study. While the program is relatively efficient, its use is rather expensive -- analysis of 100 securities costs about $300 (a case involving 96 securities required 33 minutes on an IBM 7090 computer); moreover, no more than 253 securities can be analyzed with the code. Although new programs which are being prepared specifically for the portfolio-analysis problem[7] will

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[6]The RAND code is described by Leola Cutler, Arthur E. Speckhard, and Philip Wolfe in "The RAND Quadratic Programming Code" (The RAND Corporation, Santa Monica, California, August, 1960) (mimeographed). The manner in which the code is applied to the portfolio-analysis problem is given by Neil R. Paine in "Mathematical Programming in Portfolio Selection" (Unpublished Ph.D. dissertation, School of Business Administration, University of Texas), pp. 92-95.

[7]Such a program is being written for the International Business Machines Corporation (letter from Robert T. Mertz, Data Systems Division, International Business Machines Corporation, May 12, 1961).

undoubtedly bring lower costs and greater capacity, portfolio analysis using a program which must provide for a full variance-covariance matrix will continue to be expensive and limited in capacity.

As indicated in the previous section, the diagonal model can be formulated in a manner which leaves only the diagonal of the variance-covariance matrix non-zero. This allows a considerable reduction in the computation required for solving the portfolio-analysis problem (thus decreasing machine time and cost), and also greatly reduces the amount of data which must be stored in the computer (thus increasing the maximum number of securities which can be analyzed). In order to take full advantage of these characteristics of the diagonal model, a special-purpose portfolio-analysis code was written in the Fortran language for use with IBM computing equipment.[8] The derivation of the solution technique, which uses Markowitz's critical line method,[9] is given in Appendix A; the program itself, with required inputs, is shown in Appendix B. The advantages of the code are considerable: analysis of 100 securities costs only $5 (the 96-security case which required 33 minutes with the RAND QP Code was completed in 30 seconds with this code) and as many as 2000 securities can be analyzed.

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[8] The Fortran language is described by the International Business Machines Corporation in their Reference Manual: 709 Fortran Automatic Coding System (New York: International Business Machines Corporation, 1959).

[9] Markowitz, Portfolio Selection, Appendix.

It is interesting to note that the possibility of such a substantial reduction in cost was foreseen by Miller in his review of Markowitz's book. After discussing the concept of semi-variance he suggests:

> Of greater immediate promise for those concerned with concrete application is the suggestion in chapter iv to obtain the necessary covariance inputs not by estimating each separately (which would involve n(n-1)/2 such calculations) but by deriving them indirectly from only n regressions of each security on a single "index" return. This suggestion, which can be further refined in a variety of useful ways, undoubtedly will help to break through what has been one of the major cost barriers to the use of the E-V approach.[10]

Having accomplished this anticipated reduction in cost, we are ready to investigate the value of the diagonal model in various applications.

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[10] Merton H. Miller, Review of Portfolio Selection, Efficient Diversification of Investments, by Harry M. Markowitz, The Journal of Business, The Graduate School of Business of the University of Chicago, XXXIII (October, 1960), 391-393.

III. PORTFOLIO ANALYSIS USING OBJECTIVE PREDICTION TECHNIQUES

A. Alternative Prediction Techniques

We have defined objective prediction techniques as procedures for obtaining predictions concerning security performance by applying clearly specified rules to a set of data. In the Markowitz formulation, the goal of any technique of security analysis is a set of estimates of the expected values, variances, and covariances, of the yields of the securities analyzed. One very simple objective prediction technique would merely use, for these estimates, the corresponding values in a previous period; we will call this the historical technique. An alternative objective prediction method would use the past performance of securities to obtain estimates of the parameters of the diagonal model; this will be termed the diagonal prediction technique. This chapter describes a number of tests of these two methods.

B. The Sample of Industrial Common Stocks

In order to test the alternative prediction techniques described above, 96 securities were chosen randomly from among the common stocks of industrial corporations listed on the New York Stock Exchange.[1] The yield of each security was calculated for

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[1] The sample was originally collected by Professor A.A. Alchian for use in another study. Alchian selected the securities from the population of industrial common shares which were listed in 1940 and which still existed in 1952. He also obtained the yield data for 1940-52, while the author calculated those for the period 1953-59.

every year from 1940 through 1959, using the following formula for the yield in a given year (t):[2]

$$Y_t = \frac{(\text{Price at end of Year t}) + (\text{Dividends paid during Year t})}{(\text{Price at beginning of Year t})}$$

The securities included in the sample are listed in Appendix C.

C. Estimation of the Parameters of the Diagonal Model

In order to test the performance of portfolios based on objective prediction techniques, the period 1940-59 was divided into two subperiods -- 1940-51 and 1952-59 -- so that predictions based on the performance of securities during the earlier period could be tested with data from the latter period.[3]

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[2]For actual computation of yields, a number of conventions had to be adopted to handle the large number of cases not covered explicitly by the formula. Acquisitions, mergers, etc. were handled in accordance with the convention that the investor wished to hold a single security as similar to his original holding as possible. Stock dividends of the security already held were retained while non-cash dividends representing other securities were sold at the earliest possible date. Warrants to purchase the security already held were exercised. The closing price of the security on the New York Stock Exchange on the last day of trading in Year (t) was used for the price at the end of Year (t). The price at the beginning of Year (t) was measured by the closing price of the security on the New York Stock Exchange on the last day of Year (t-1). Dividends paid during Year (t) were assumed to equal those paid to holders of record throughout Year (t). In a few cases, of course, the data required to calculate the yields in strict accordance with the adopted conventions were not available and some values had to be estimated. Major sources of the data were: Moody's Industrial Manual, The New York Times, The Bank and Quotation Record, and The Commercial and Financial Chronicle.

[3]These two subperiods were selected on arbitrary grounds, but they were remarkably similar. The average annual yield of an index

Computation of the parameters for the historical prediction technique was straightforward: the average yield of a security during the period 1940-51 served as an estimate of E_i, the variance of its yield around that average as an estimate of C_{ii}, and the covariances between its yield and those of each of the other securities as estimates of the C_{ij} terms.

The first decision required for estimating the parameters of the diagonal model concerned the appropriate measure for the market index. For this test an unweighted average of the yields of the 96 securities in the sample was selected as the measure of (I); then a regression line of the form:

$$Y_i = A_i + B_i \cdot I$$

was fitted between the yield of each security and the level of (I) using the least-squares method, with (I) considered as the independent variable. This line provided the estimates of A_i and B_i for the prediction technique. Q_i, the variance of the random component w_i was computed by subtracting the variance due to the market from the total variance of a security:

$$Q_i = V_i - (B_i)^2 \cdot V_I$$

where V_i is the total variance of the security and V_I is the

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of the 96 securities was 1.185 in both periods, while the standard deviation of its yield was .233 in the earlier period and .250 in the latter.

variance of the index (I). The two remaining parameters were easily determined: the average value of (I) was used for A_{n+1} and its variance V_I for Q_{n+1}.

The extent of correlation between the individual securities and the market index is indicated in Fig. 1, which shows the distribution of the correlation coefficients between the yield of each security and that of (I). The coefficients were computed according to the formula:

$$CC_{i,I} = \sqrt{\frac{(B_i)^2 \cdot V_I}{V_i}}$$

The median value was slightly above +0.7, which is significant at the 98% confidence level. Over 75% of the securities had correlation coefficients which were significant at or above the 95% confidence level. Figure 2 shows the distribution of the values of (B_i) -- the responsiveness of the yields of the securities to the market index; the median value was approximately .90.

D. Comparison of the Techniques

The two prediction techniques described differ significantly in the number of parameters required to estimate the performance of securities. The historical technique requires 4752 parameters for 96 securities: 96 expected values, 96 variances, and 4560 covariances. The diagonal model uses only 290 parameters for this number of securities: 96 values of A_i, 96 values of B_i, 96 values

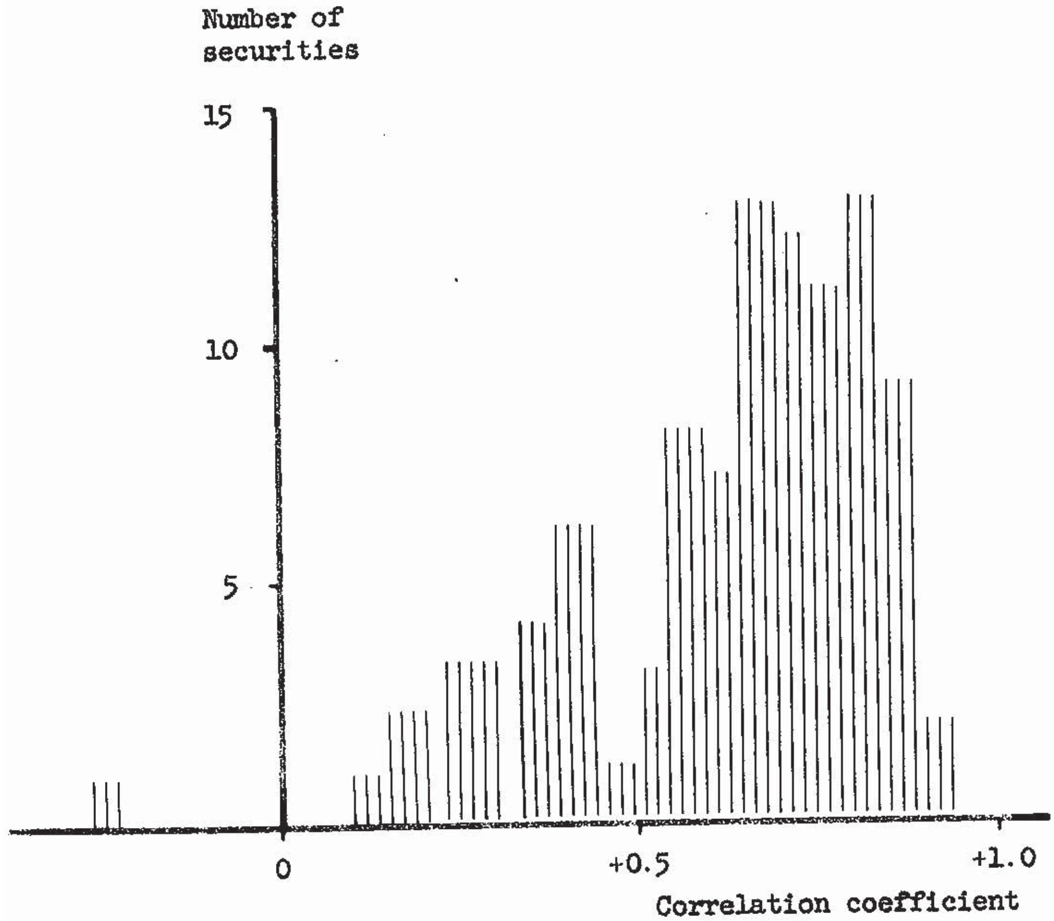

Fig. 1. - Correlation coefficients between yields of individual securities and yield of the market in general: 96 industrial common stocks, 1940-1951

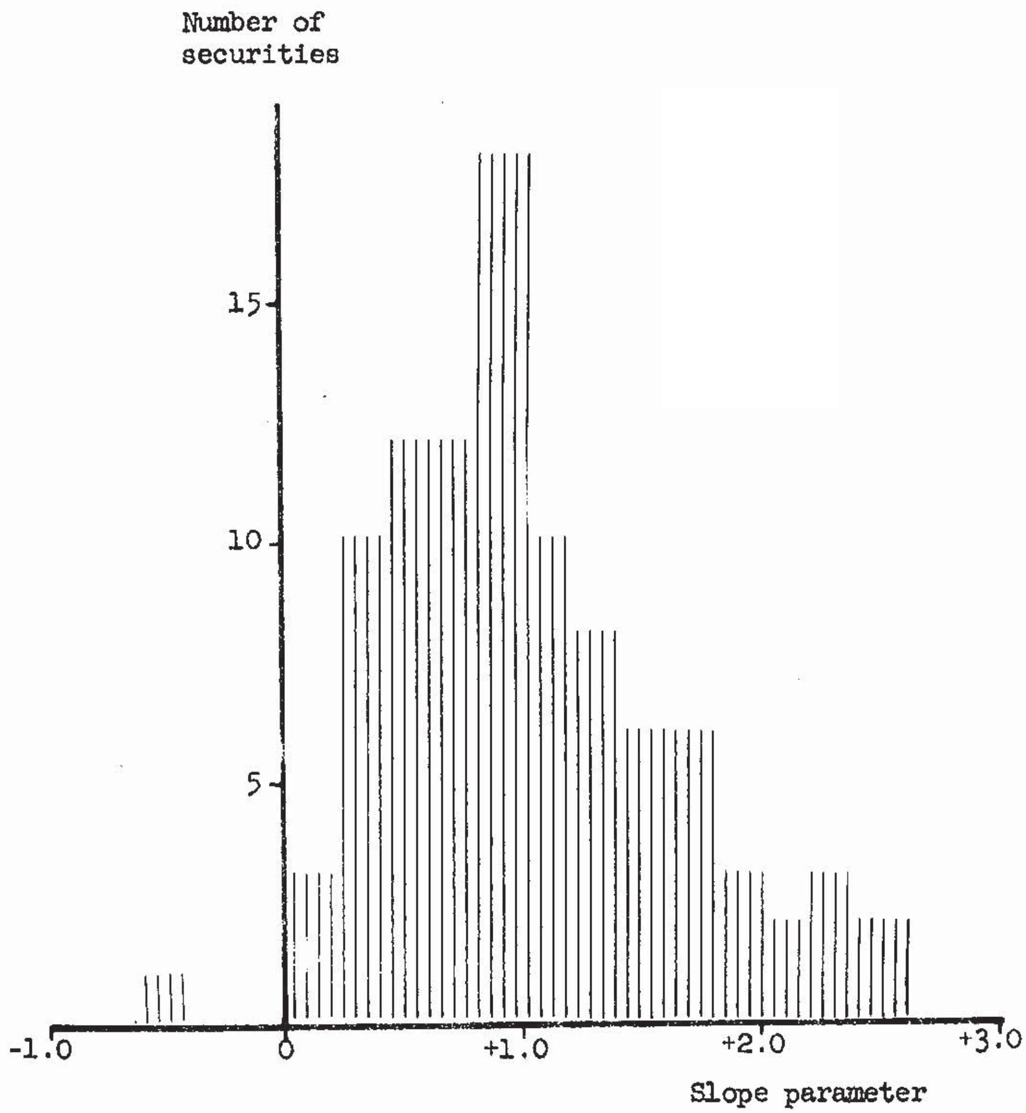

Fig. 2. - Slope parameters from regression of yields of individual securities on yield of the market in general: 96 industrial common stocks, 1940-1951

of Q_i, the value of A_{n+1}, and that of Q_{n+1}. The manner in which the parameters for the diagonal technique were estimated insured that the two methods gave equal estimates for the 96 expected values and the 96 variance terms; all differences occurred in the 4560 covariance terms.

The difference between a covariance term implied by the diagonal technique ($cov^d_{i,j}$) and the actual value ($cov^a_{i,j}$) can be made more meaningful if it is divided by the product of the standard deviations of the two securities in order to indicate the relative magnitude of the error. Let (d) represent this relative error; then:

$$d \equiv \frac{cov^d_{i,j} - cov^a_{i,j}}{\sigma_i \sigma_j}$$

Since the correlation coefficient between two securities is merely the covariance divided by the product of the standard deviations, this term can be considered as the difference between the correlation coefficient implied by the diagonal technique and that implied by the historical technique. The distribution of the values of (d) for the 4560 combinations of the 96 securities is shown in Fig. 3. The standard deviation of (d) is .189; more important, its average value is +.003, indicating that the diagonal prediction technique gives virtually unbiased estimates of the covariance terms.

Although Fig. 3 indicates that the differences between the covariance terms of the two prediction techniques are relatively small, those which do exist may be very important. Further

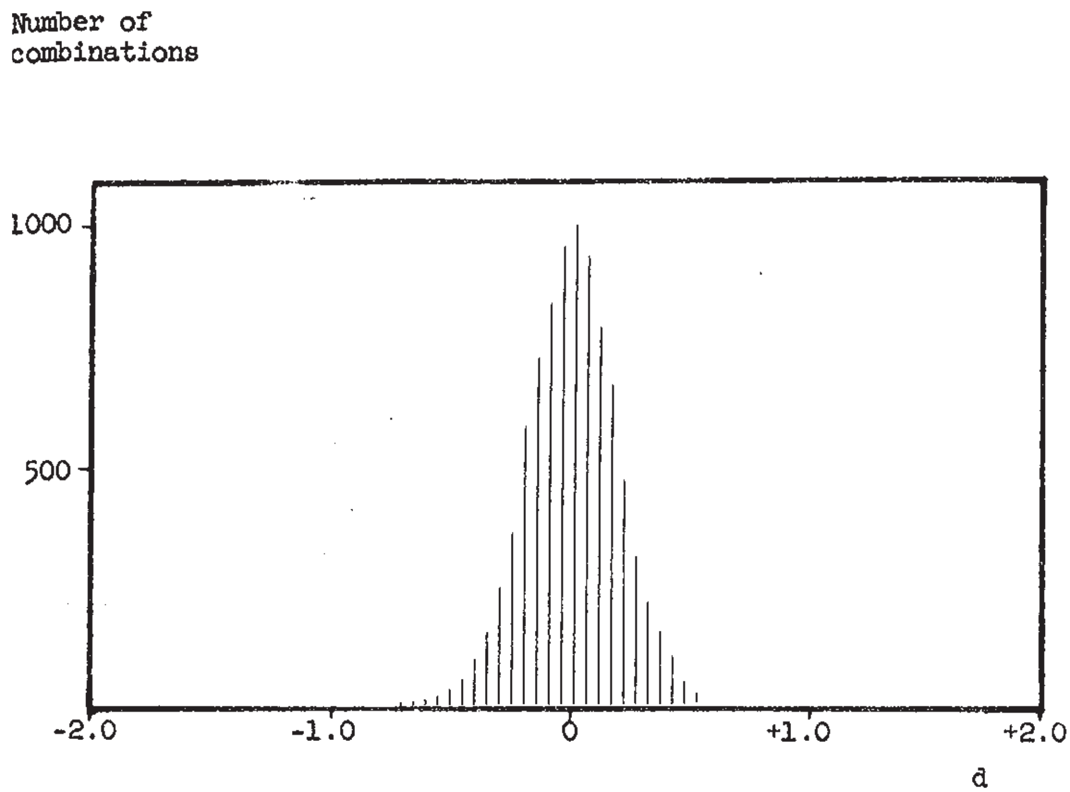

Fig. 3. - Distribution of (d): 4,560 combinations of industrial common stocks, 1940-1951

evidence is required before the relative desirability of the two techniques can be completely assessed. A number of aspects of the two methods can be compared: the predicted performance of efficient portfolios generated by the associated portfolio analyses, the composition of those portfolios, and, most important, their actual performance in some subsequent period.

To make these comparisons, two subsamples, each containing 20 securities, were drawn randomly from the full sample of 96 securities. For each sample, the parameters of both prediction techniques were estimated from the performance of the securities during the period 1940-51, portfolio analyses were performed, and the performance of the resulting portfolios was evaluated over the period 1952-59.

The first comparison concerns the predicted performance of efficient portfolios based on the two techniques. Figure 4 shows the predicted combinations of average yield and standard deviation of yield for efficient portfolios chosen with the two techniques from the securities in Sample 1; figure 5 illustrates comparable data for portfolios selected from the securities in Sample 2. Quite obsiously, the two techniques differ only slightly in their predictions of attainable combinations of the two major elements of portfolio performance. The predicted performance of efficient portfolios selected by the historical technique from the first sample is slightly better than that of portfolios chosen with the diagonal technique in the lower range of expected yield, but

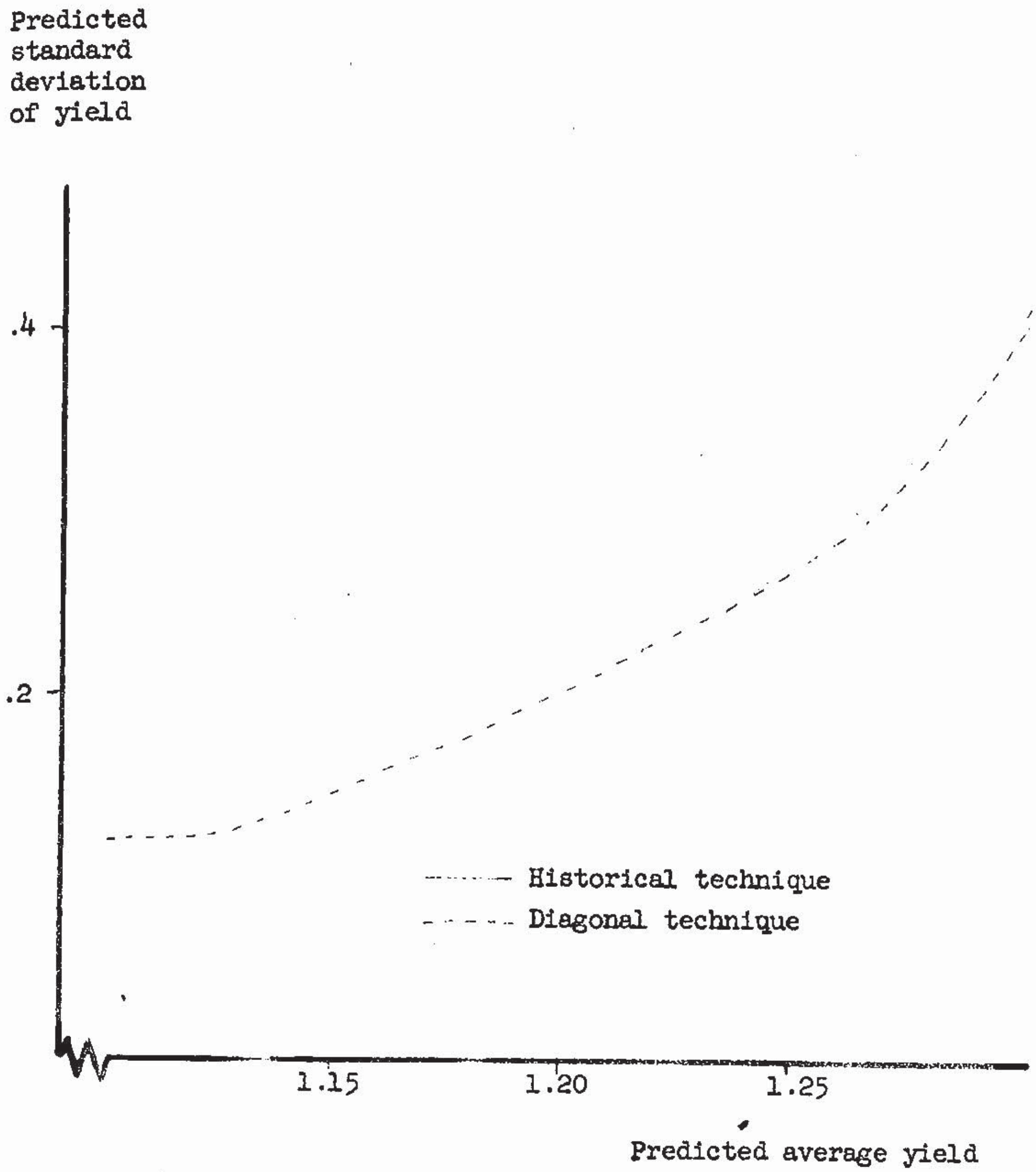

Fig. 4. - Predicted values of average yield and standard deviation of yield for efficient portfolios: twenty-security sample 1

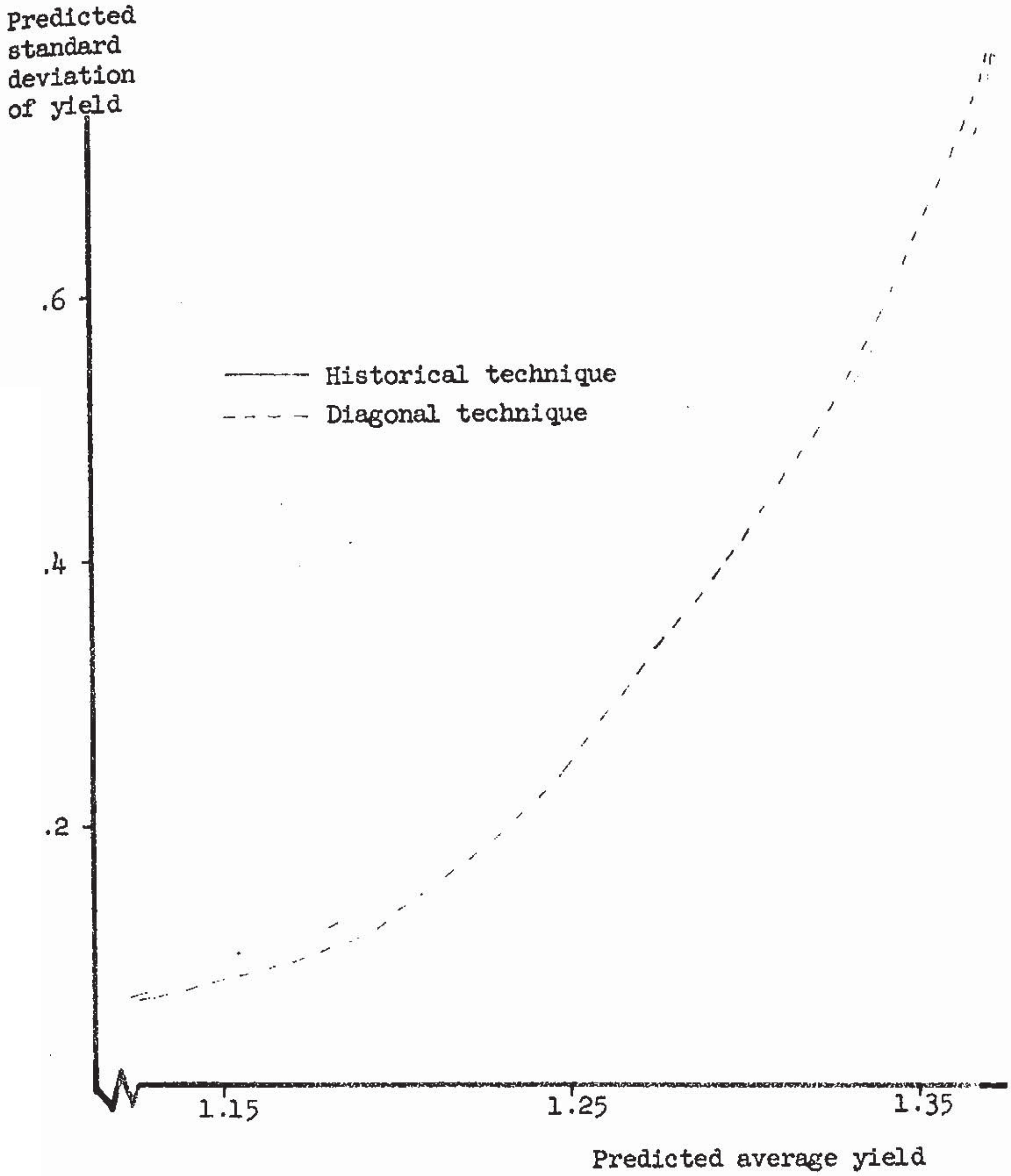

Fig. 5. - Predicted values of average yield and standard deviation of yield for efficient portfolios: twenty-security sample 2

slightly worse in the range of higher yields. These relationships were reversed when the methods were applied to Sample 2. It appears that the techniques do not differ significantly with regard to this attribute.

A second aspect of portfolios selected by the two methods concerns the securities included. To what extent do the two techniques select the same securities? Figures 6 through 9 display the composition of efficient portfolios selected with the two techniques. In each figure the horizontal axis indicates a predicted average yield, while the vertical axis indicates the amount of the corresponding efficient portfolio invested in each of the 20 securities. Thus, Fig. 6 indicates that the value of the efficient portfolio with a predicted average yield of 1.15 is divided among the securities in the following manner:

Security	Value of Portfolio
10	0.06
15	0.10
8	0.15
20	0.32
6	0.35
Not shown[4]	0.02
	1.00

Examination of the figures indicates that the two prediction techniques are remarkably similar with regard to the portfolios selected: the differences between Figs. 6 and 7 are slight, as are

[4] In order to simplify the presentation, securities appearing small amounts are not shown in Figs. 6 through 9.

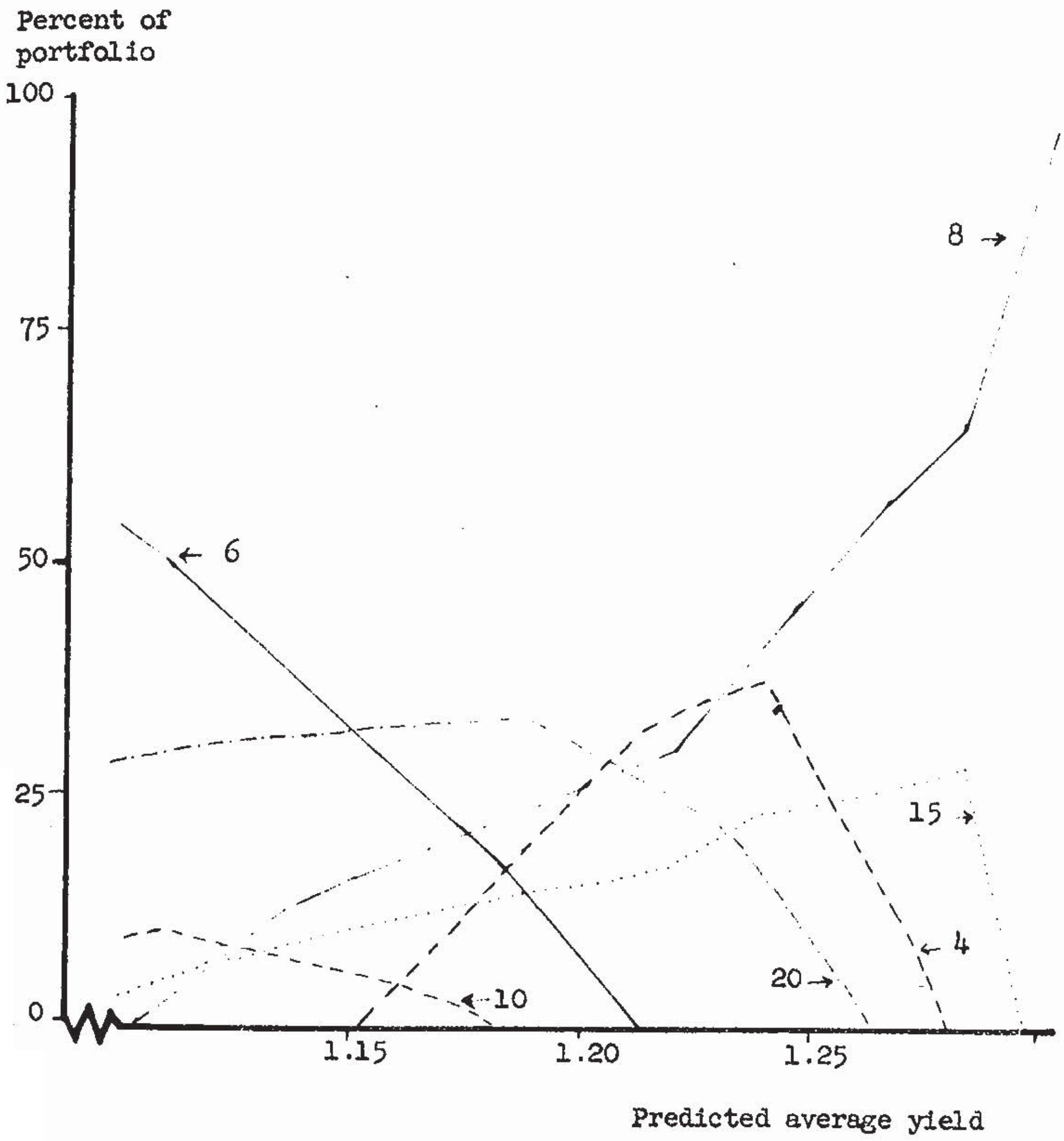

Fig. 6. - Composition of efficient portfolios: historical technique, twenty-security sample 1

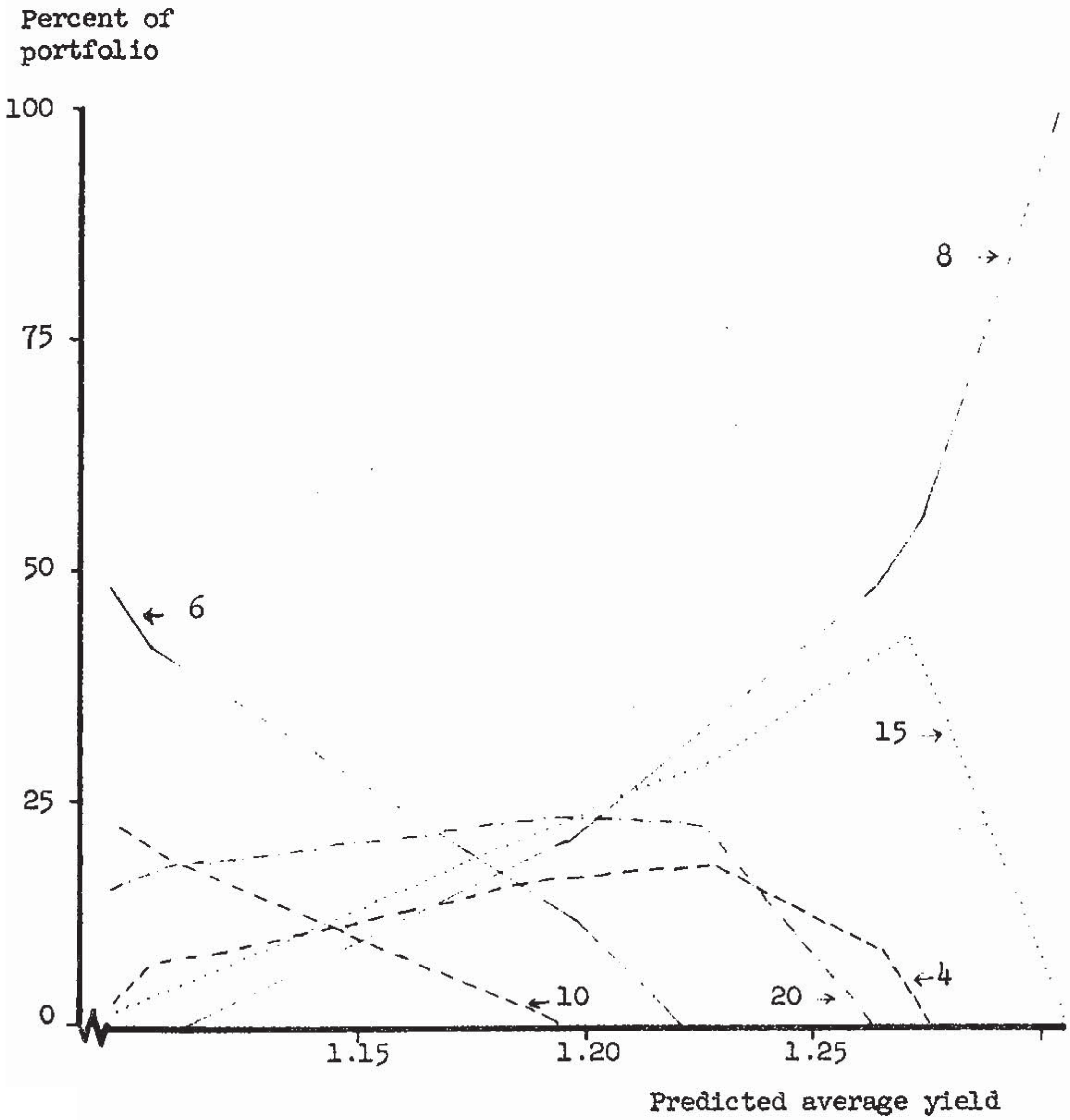

Fig. 7. - Composition of efficient portfolios: diagonal technique, twenty-security sample 1

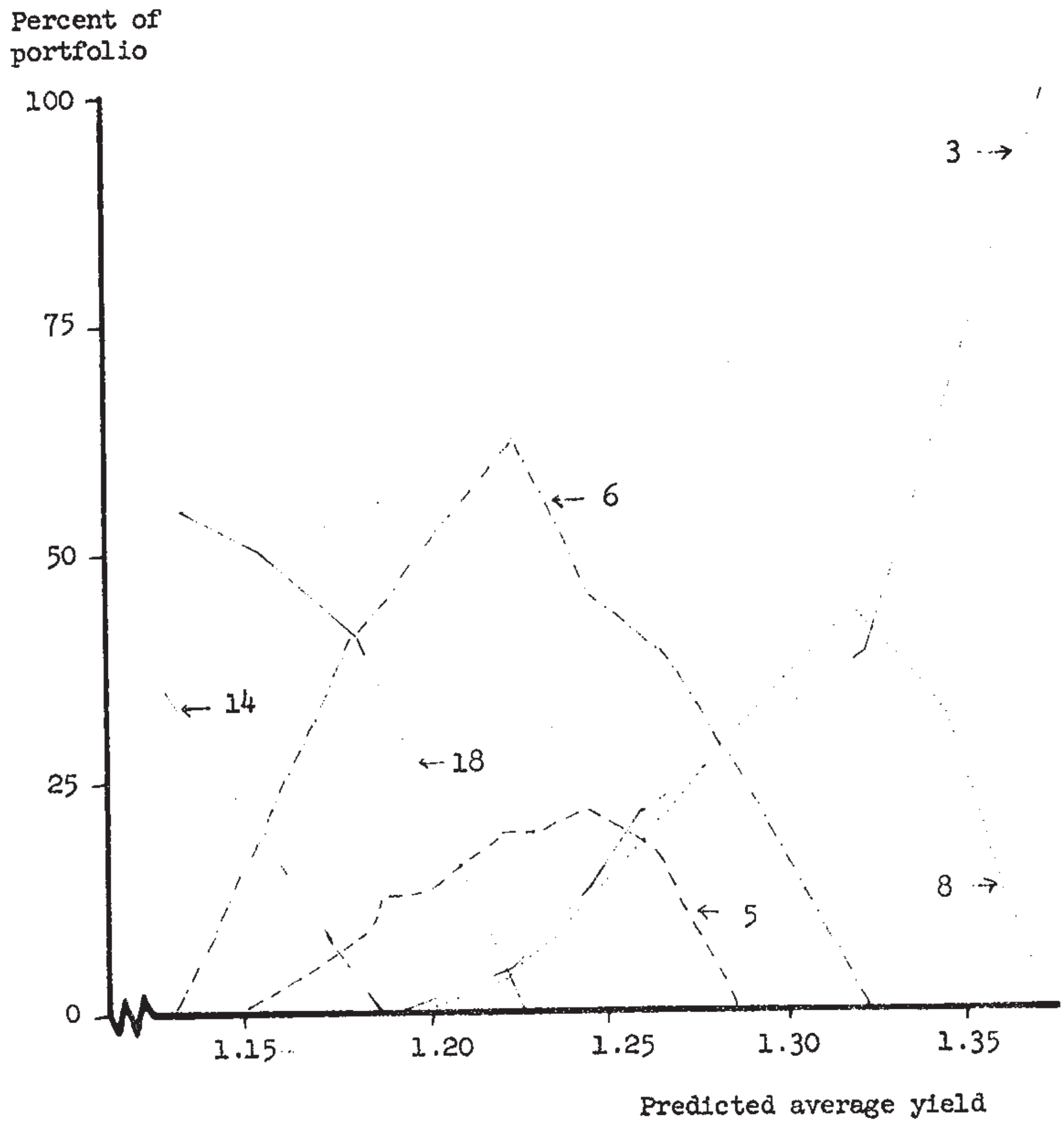

Fig. 8. - Composition of efficient portfolios: <u>historical technique</u>, twenty-security sample 2

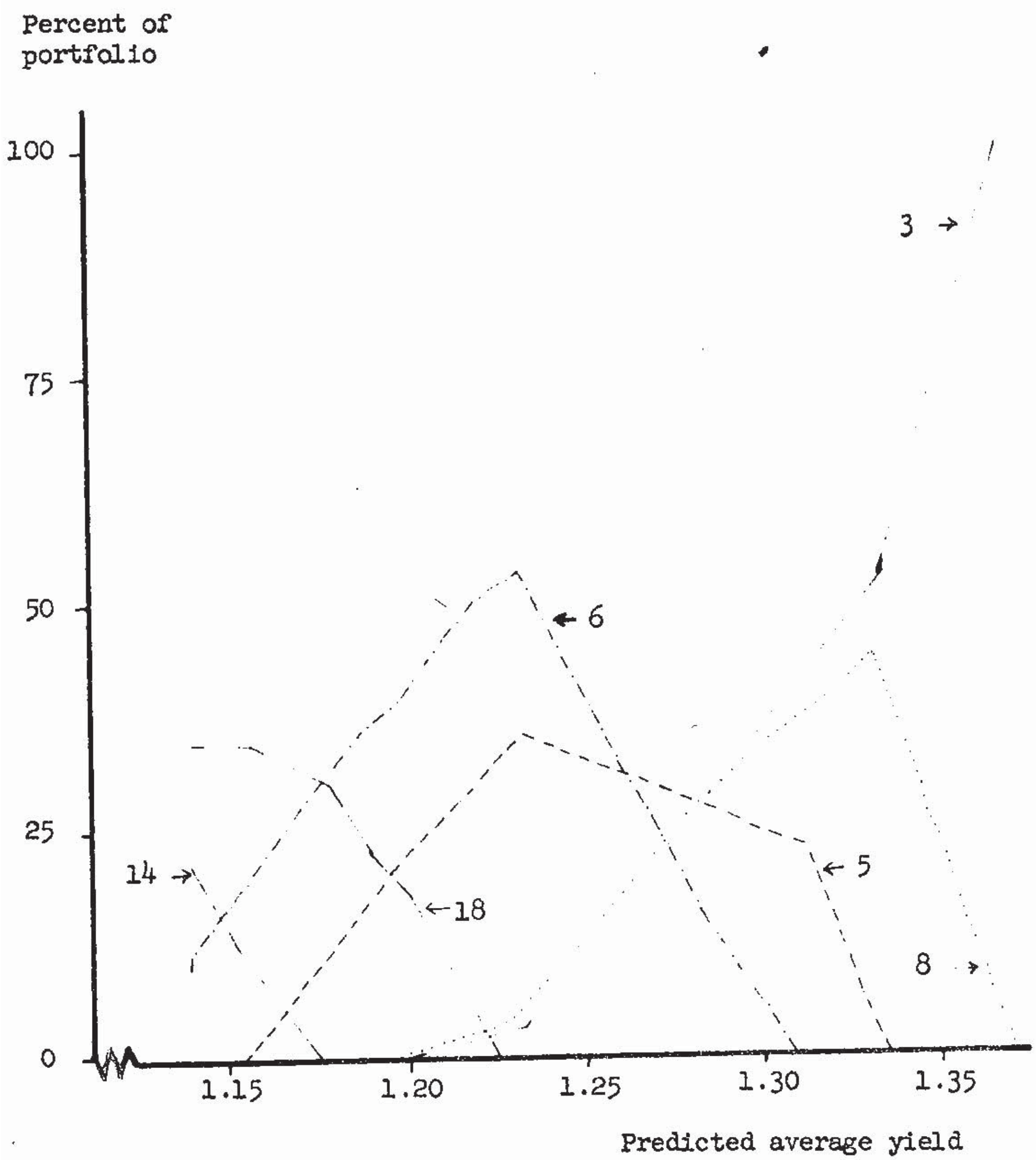

Fig. 9. - Composition of efficient portfolios: diagonal technique, twenty-security sample 2

those between Figs. 8 and 9. The results of this comparison are similar to those of the previous test: there appears to be no significant difference between the techniques.

The final comparisons of these prediction techniques concern the performance of efficient portfolios selected in the associated portfolio analyses. Although the methods lead to portfolios which are quite similar, it is possible that the small remaining differences in composition could lead to significant differences in actual performance. To investigate this possibility, the yields of the securities during the period 1952-59 were used to determine the average yield and standard deviation of yield associated with each of the efficient portfolios selected by the two prediction techniques. We will treat these two aspects of performance separately. In each case, two separate comparisons will be made: the first will investigate the relative performance of comparable portfolios selected by the two techniques, while the second will compare the relative degree of correspondence between predictions and actual outcomes associated with the two techniques.

Figure 10 shows the relationship between predicted average yield and actual average yield for the efficient portfolios selected with the two techniques from the securities in Sample 1. In this case, the historical technique was obviously superior: all portfolios chosen with this technique did at least as well as comparable portfolios (those with equal predicted average yields) selected with the diagonal technique. This relationship did not

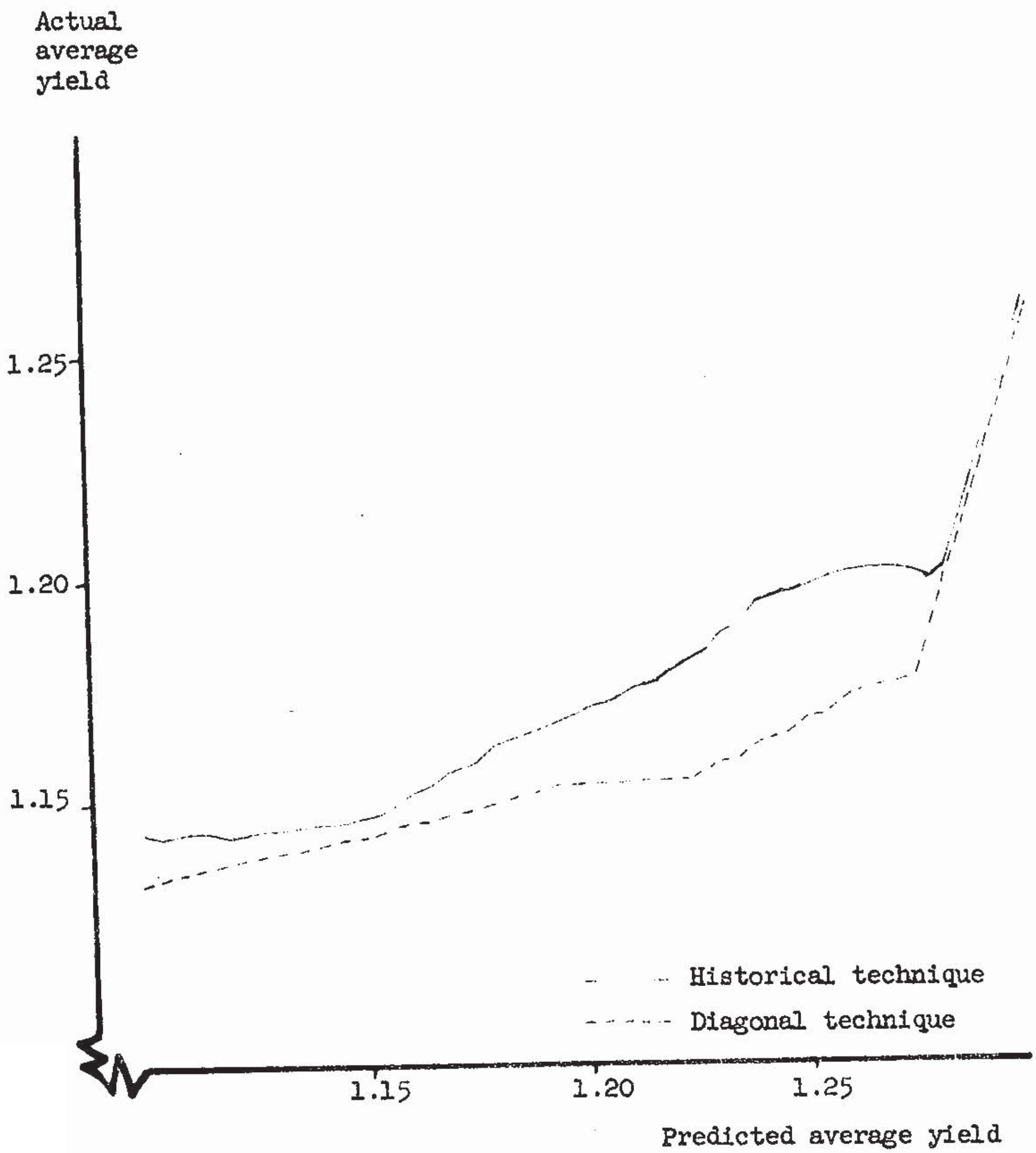

Fig. 10. - Predicted average yields and actual average yields 1952-1959 for efficient portfolios: twenty-security sample 1

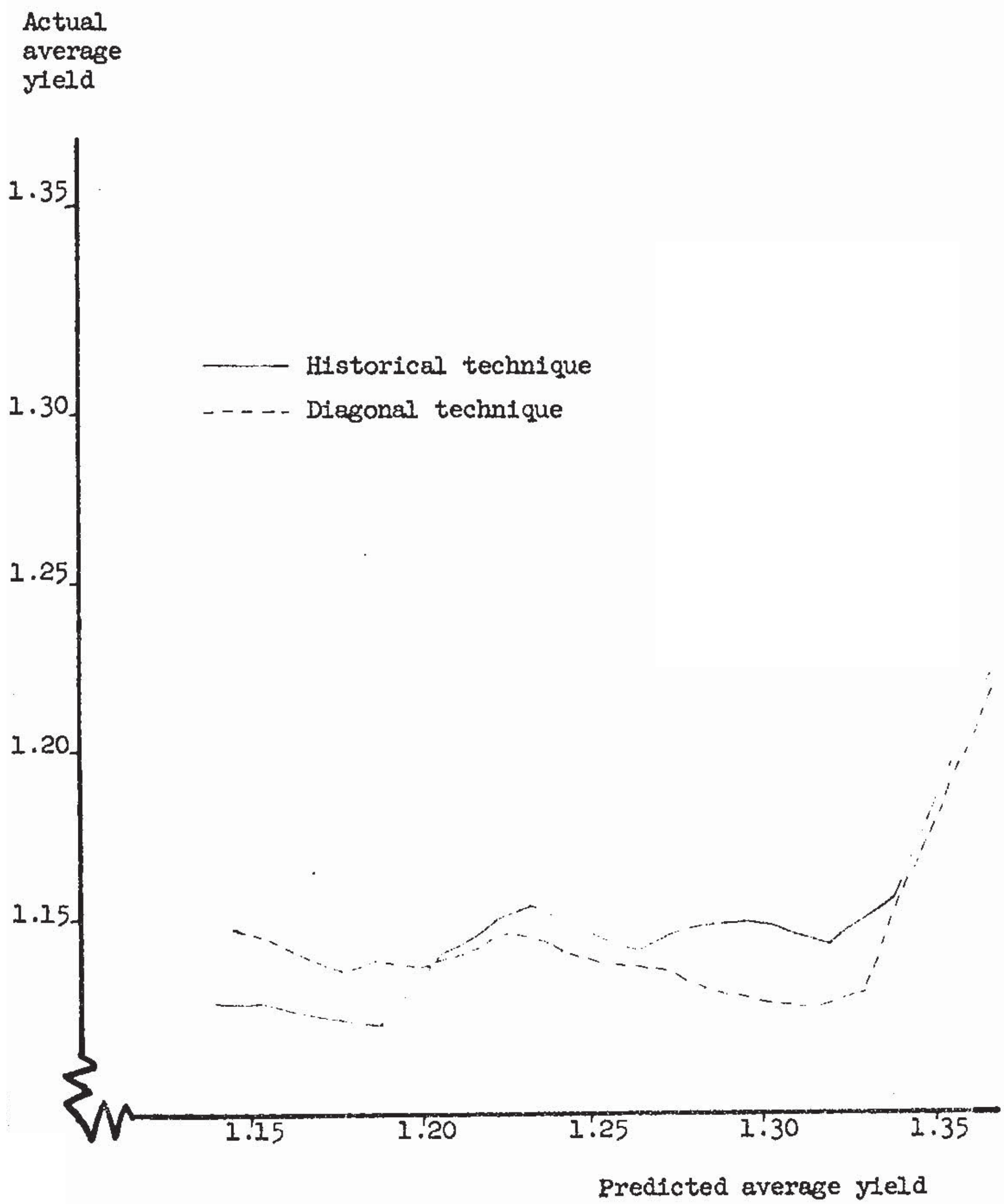

Fig. 11. - Predicted average yields and actual average yields 1952-1959 for efficient portfolios: twenty-security sample 2

hold, however, in the analysis of Sample 2. As shown in Fig. 11, the historical technique gave inferior portfolios in the range of lower predicted yields and superior portfolios in the range of higher yields. It appears that neither technique is obviously superior in terms of average yields actually attained by the portfolios selected in the associated portfolio analyses.

Figures 10 and 11 illustrate not only the relative performance of the two sets of portfolios in terms of average yields but also the relationship between predicted average yield and actual yield for each technique. A somewhat more explicit indication of this relationship can be obtained with some measure of rank correlation, although such measures must be used with caution. In order to indicate the degree of consistency between a ranking of portfolios based on predicted average yields and one based on actual average yields, a group of portfolios was selected from each set of efficient portfolios,[5] and the value of Spearman's coefficient of rank correlation[6] was computed; Table 1 shows the results.

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[5]The portfolios were selected on the basis of predicted average yields, starting with the more conservative and using an interval of .005; thus, from the analysis of Sample 1 with the diagonal technique, 42 portfolios were selected with predicted average yields of 1.095, 1.100, 1.105, etc., through 1.300.

[6]Let (d_i) be the difference between the rank of portfolio i based on predicted values and that based on actual values, and N the number of securities. Spearman's coefficient is:

$$R_r = 1 - \frac{6 \sum_{i=1}^{N} (d_i)^2}{N^3 - N}$$

Since N exceeds 25 for the samples which we are investigating, the

Table 1

VALUES OF SPEARMAN'S RANK CORRELATION COEFFICIENT FOR COMPARISONS BETWEEN PREDICTED AND ACTUAL AVERAGE YIELD

Sample	Portfolios Selected with the Historical Technique	Portfolios Selected with the Diagonal Technique
1	0.993	0.999
2	0.748	-0.125

Both techniques give remarkably good results with Sample 1: both rank correlation coefficients are more than six standard deviations above zero. Results with Sample 2 are not as satisfactory; the portfolios selected with the historical technique have a coefficient slightly more than five standard deviations above zero, but those chosen with the diagonal technique actually exhibit a slight negative correlation, as can be seen in Fig. 11 from the generally negative slope of the corresponding curve.

Comparisons of the two techniques with respect to standard deviations of yield can be briefly presented since the analysis is similar to that used for average yields. Figures 12 and 13 display

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null hypothesis can be tested using a standard deviation computed as follows:

$$S_{R_r} = \frac{1}{\sqrt{N - 1}}$$

For a more complete discussion of Spearman's coefficient, see Frederick C. Mills, Statistical Methods (3d ed. rev.; New York: Henry Holt and Company, 1955), pp. 311-317.

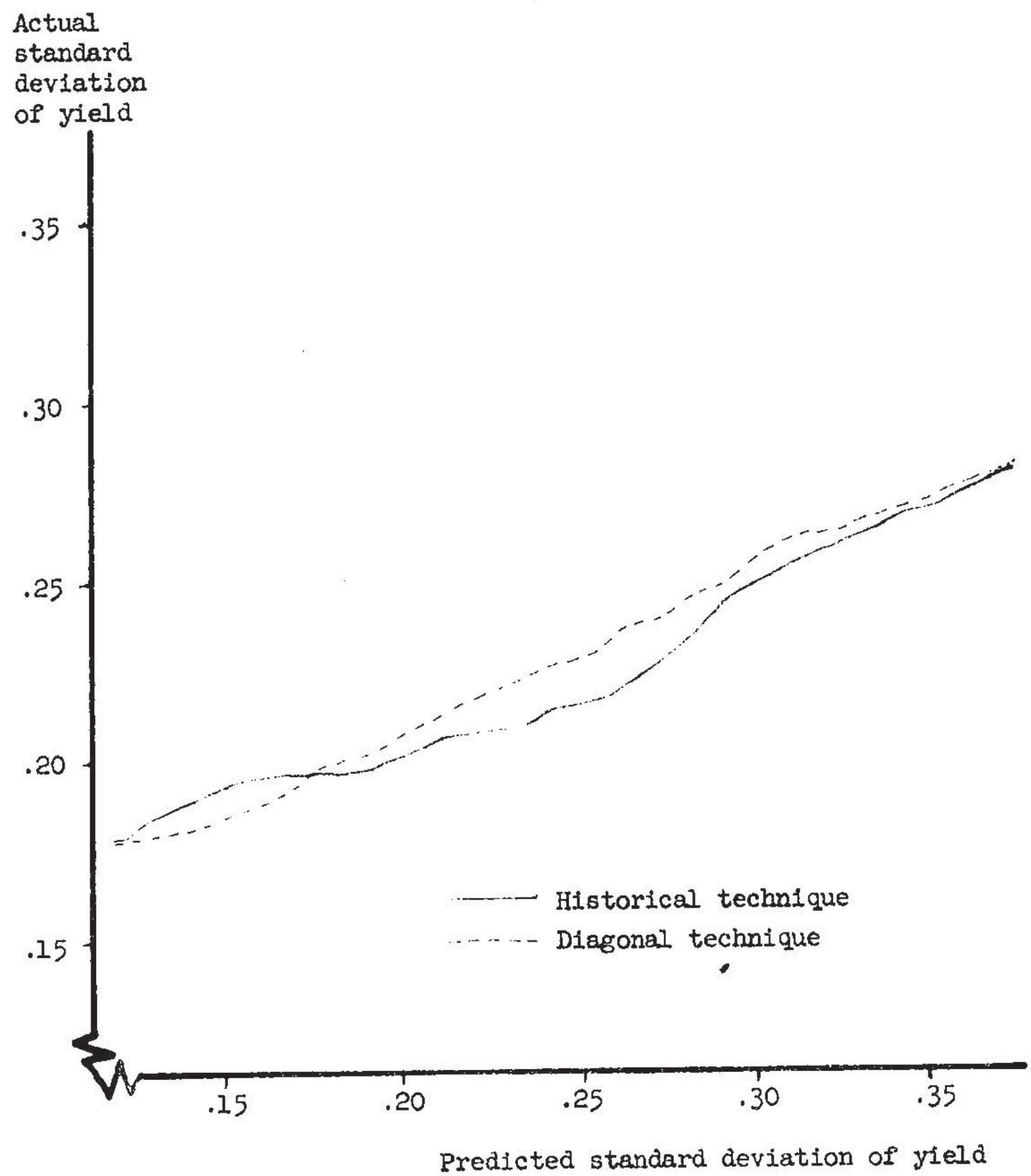

Fig. 12. - Predicted standard deviation of yield and actual standard deviation of yield 1952-1959 for efficient portfolios: twenty-security sample 1

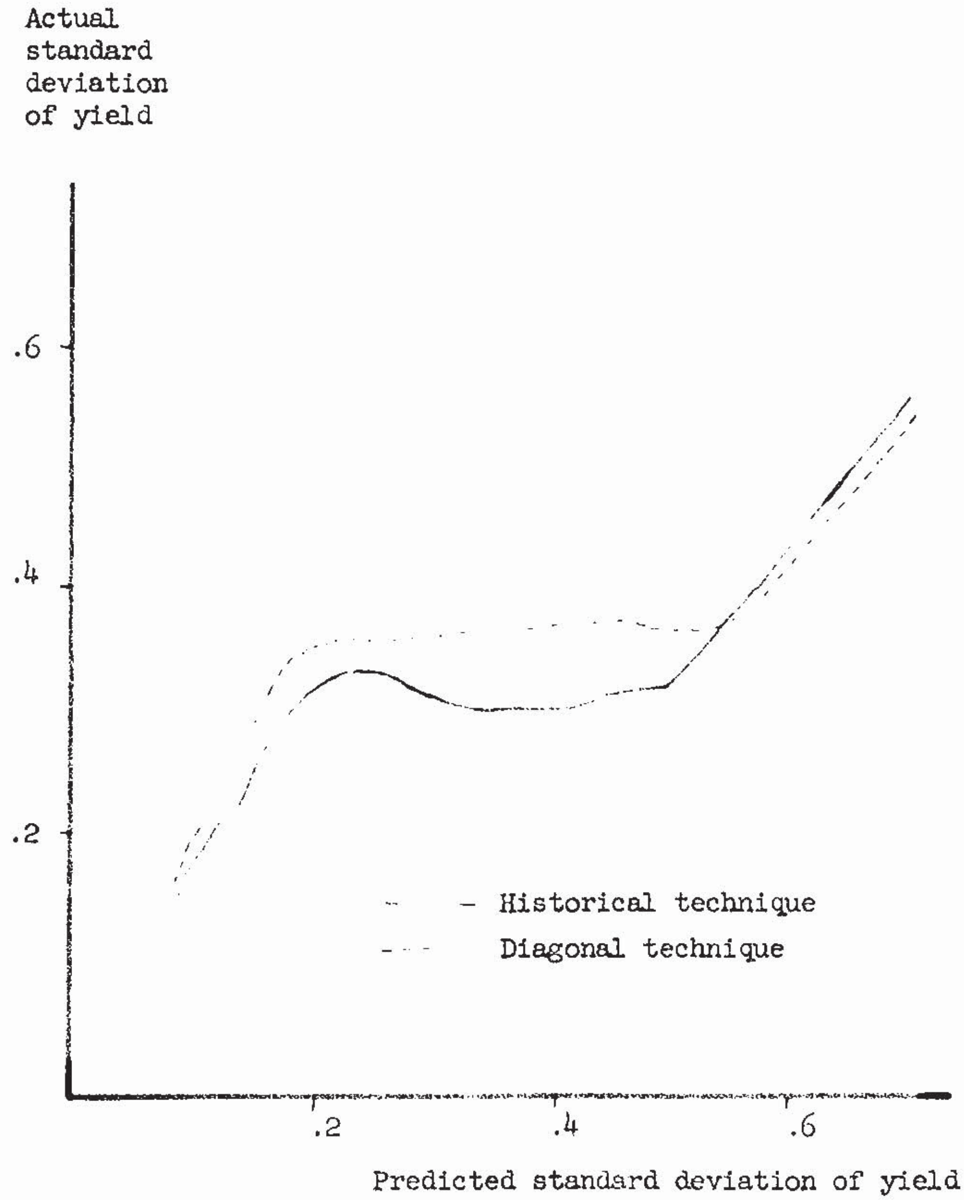

Fig. 13. - Predicted standard deviation of yield and actual standard deviation of yield 1952-1959 for efficient portfolios: twenty-security sample 2

the relationship between predicted and actual values of standard deviation of yield for the various sets of efficient portfolios. In the analysis of the securities of Sample 1, the more conservative portfolios selected with the historical technique had higher actual standard deviations than did comparable portfolios (with equal predicted standard deviations) selected with the diagonal technique; on the other hand, in the higher ranges of predicted standard deviation, the portfolios selected with the historical technique proved superior. In the analysis of the securities of Sample 2 this relationship was reversed. This comparison, like those which preceded it, fails to establish the superiority of either method.

Table 2 gives the values of Spearman's coefficient of rank correlation for comparisons of the ranking of efficient portfolios based on predicted standard deviations of yield with that based on actual standard deviations.[7]

Table 2

VALUES OF SPEARMAN'S RANK CORRELATION COEFFICIENT FOR COMPARISONS BETWEEN PREDICTED AND ACTUAL STANDARD DEVIATION OF YIELD

Sample	Portfolios Selected with the Historical Technique	Portfolios Selected with the Diagonal Technique
1	0.999	0.996
2	0.865	0.995

[7]The portfolios selected for the earlier comparisons of predicted and actual average yield were used for this test.

The two techniques performed equally well when applied to Sample 1: both coefficients are more than six standard deviations above zero. In the analysis of the securities of Sample 2, the portfolios selected with the diagonal technique gave a slightly better performance than those selected with the historical technique: the coefficient of the former was more than 6.5 standard deviations above zero, that of the latter was less than 5.9.

This completes the comparison of the two objective prediction techniques. No strong evidence has been discovered to support the argument that the historical technique is markedly superior to the diagonal technique. Since this is the case, and since the cost of portfolio analysis based on the diagonal model is so much less than that required by the historical technique, we will proceed to investigate the predictive abilities of the diagonal technique somewhat more extensively, using the full sample of 96 securities.

E. The Performance of Portfolios Selected with the Diagonal Technique

In order to provide an additional test of the diagonal prediction technique, a portfolio analysis of the 96-security sample was made, using the parameters estimated in the manner described in Sec. C of this chapter. Figure 14 shows the relationship between predicted values of average yield and actual values in the period 1952-59 for the resulting set of efficient portfolios. The correspondence between predicted and actual values of average

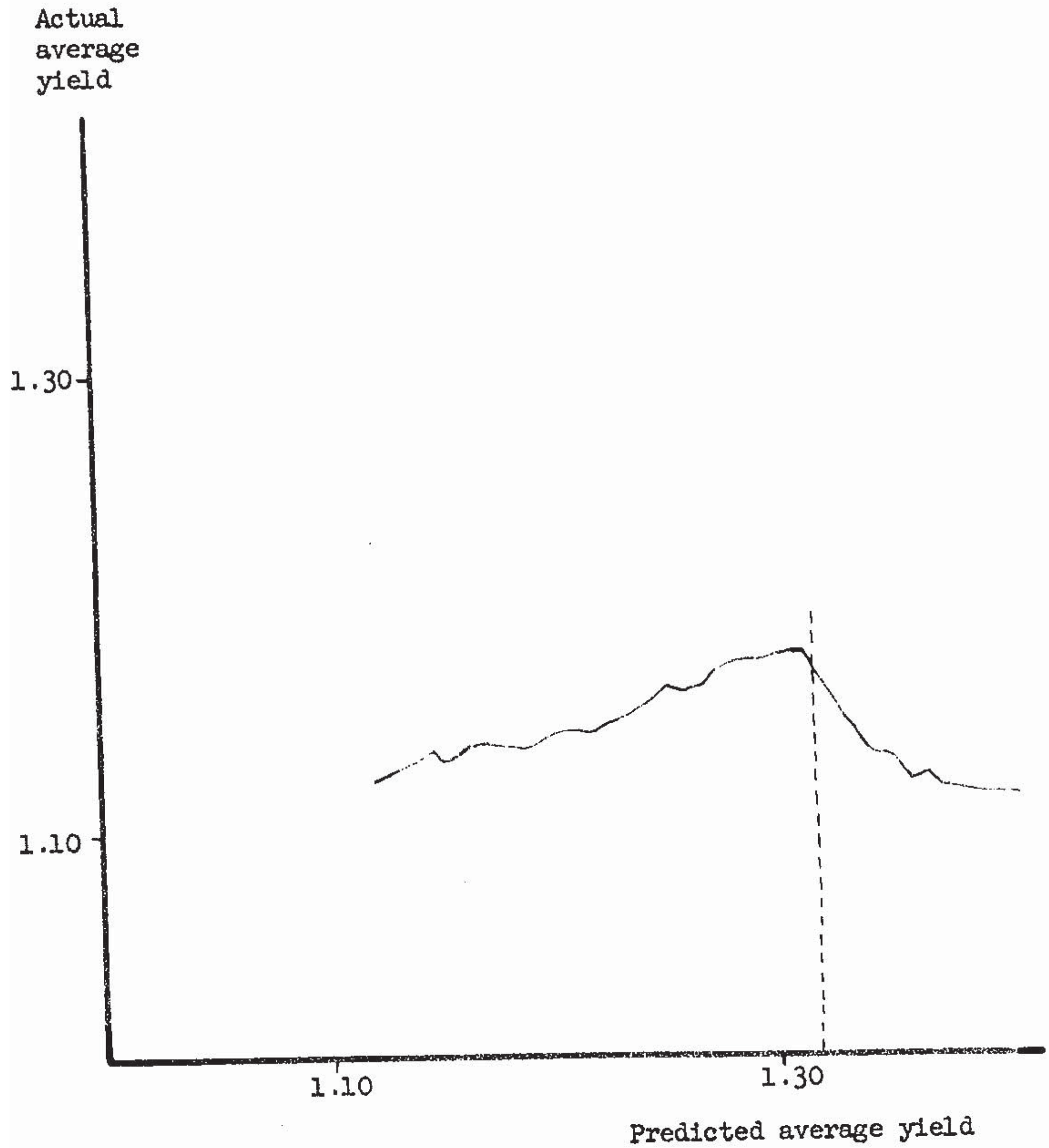

Fig. 14. - Predicted average yields and actual average yields 1952-1959 for efficient portfolios: 96-security sample

yield for a group of these portfolios[8] was measured with Spearman's rank correlation coefficient. The value of the coefficient was negative: -.162. This result, together with Fig. 14, suggests that the diagonal technique is a relatively poor predictor of average yield.

Figure 15 shows the relationship between predicted and actual values of the standard deviation of yield for the set of efficient portfolios. The correspondence between predicted and actual value appears to be much greater than with average yields. This is confirmed by the Spearman coefficient for the rankings based on standard deviations: the value is .923, almost seven standard deviations above zero.

It is apparent from both Figs. 14 and 15 that the predictive ability of the diagonal technique is much better in the range of conservative portfolios than in that of portfolios with higher risk. Since this result may be fortuitous, it would be unwise to attach too much significance to it. On the other hand, there may be a perfectly valid reason for the better predictive ability with the more conservative portfolios. Figure 16 shows the number of securities in the efficient portfolios studied. As one might expect, efficient portfolios with high predicted average yield (and standard deviation of yield) contain fewer securities than do the more conservative portfolios. With only 96 securities from

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[8] Chosen according to the rule used for the tests described in Sec. D.

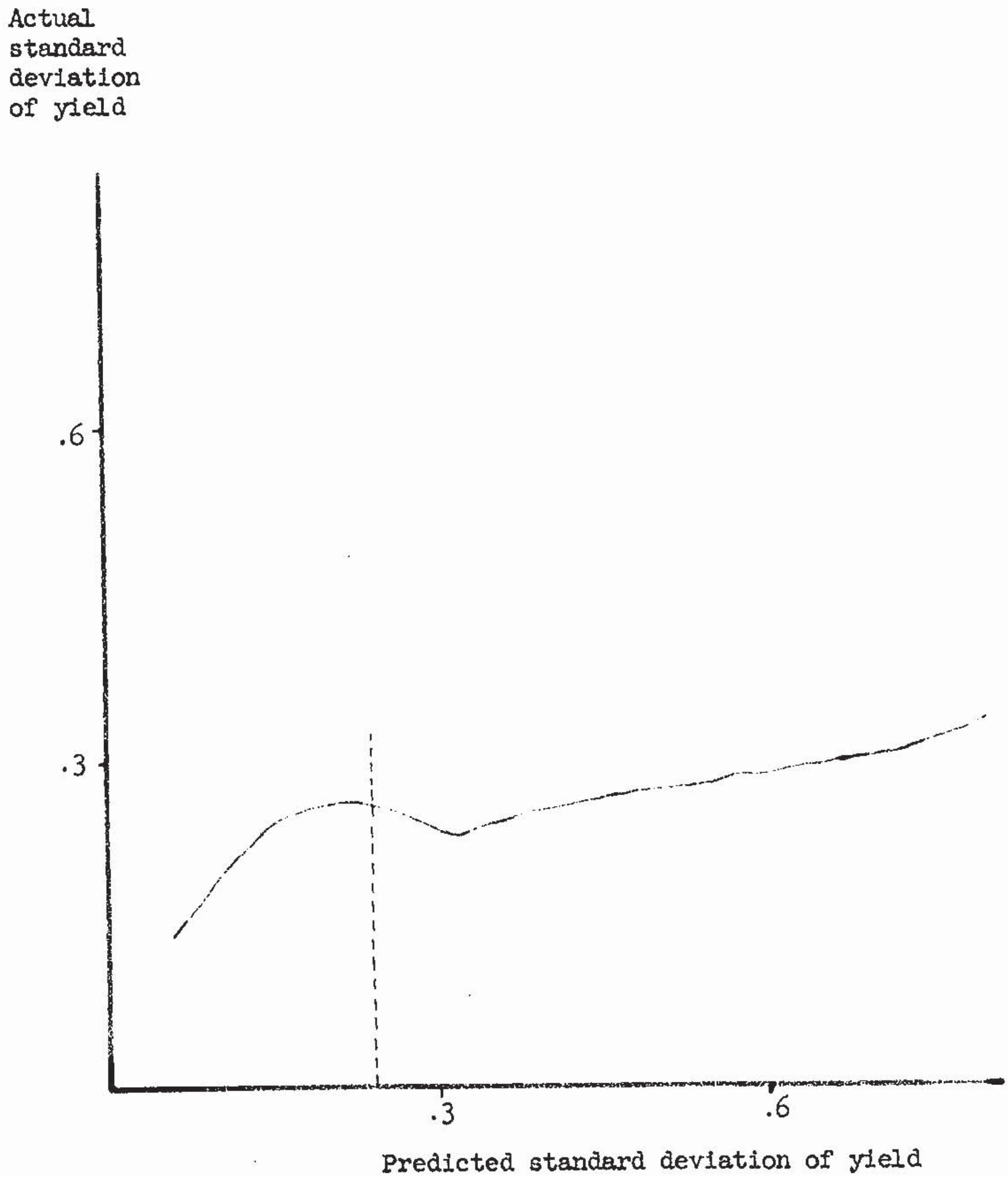

Fig. 15. - Predicted standard deviation of yield and actual standard deviation of yield 1952-1959 for efficient portfolios: 96-security sample

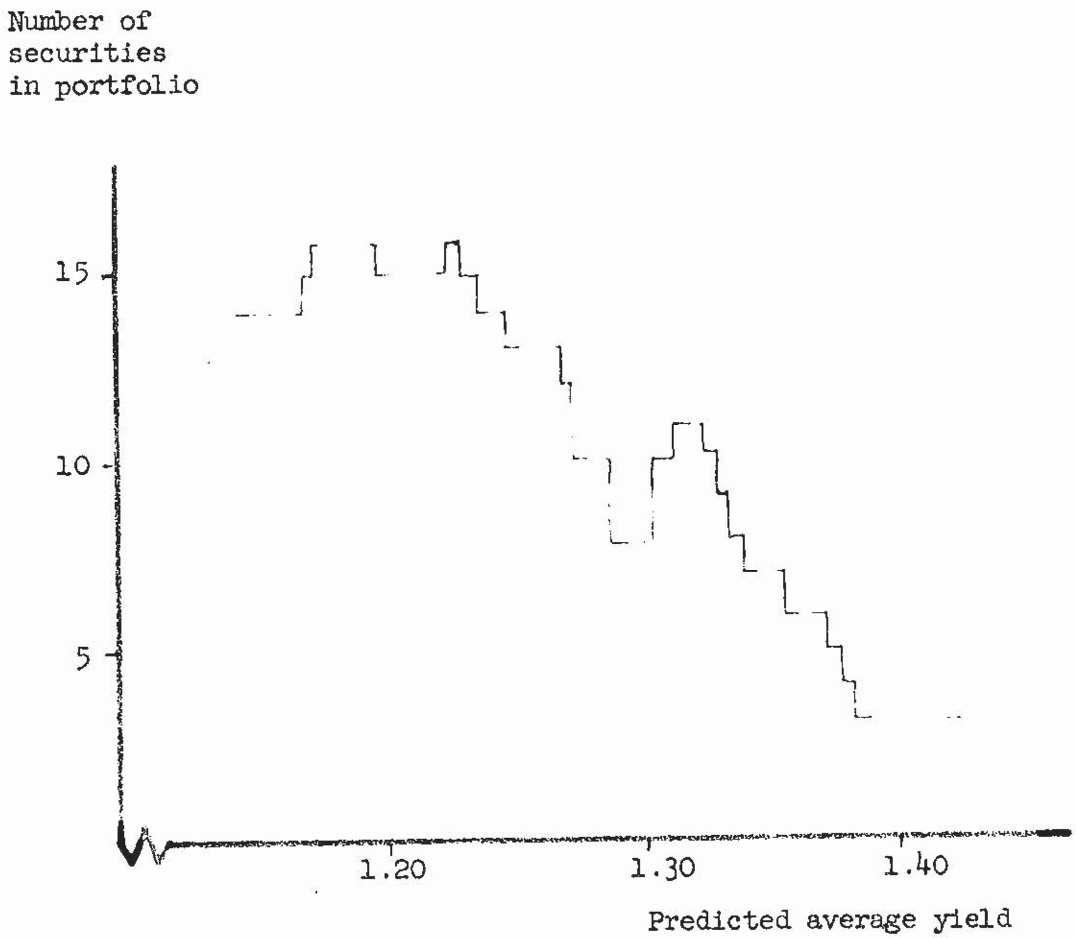

Fig. 16. - Predicted average yield and number of securities in efficient portfolios: 96-security sample

which to choose, the analysis selects portfolios with very few securities in this range of higher risk. It is not surprising that predictions based on four to eight securities should be poorer than those based on twelve to sixteen.

Figure 16 indicates that approximately the first two-thirds of the portfolios (in terms of predicted average yield) contain at least eight securities. The rank correlation coefficients for these portfolios are very high: the coefficient based on average yield is .993, that based on standard deviation of yield is .996. Both are more than seven standard deviations above zero.

It is very easy, of course, to select a range over which a technique performs well and then to rationalize the selection of that range. On the other hand, portfolios selected by the technique which contain a fairly large number of securities may be worth consideration, while those containing a small number of securities should probably be excluded on purely _a priori_ grounds. In order to differentiate between the performance of the larger portfolios and that of the smaller, we will arbitrarily divide the set into two groups: portfolios with at least eight securities, and those with less than eight. As indicated in Fig. 16, the former group lies in the range with predicted average yields of less than 1.32; for convenience, vertical lines have been drawn in both Figs. 14 and 15 to separate these two groups of portfolios.

Figure 17 shows the combinations of average yield and standard deviation of yield actually achieved by the set of efficient port-

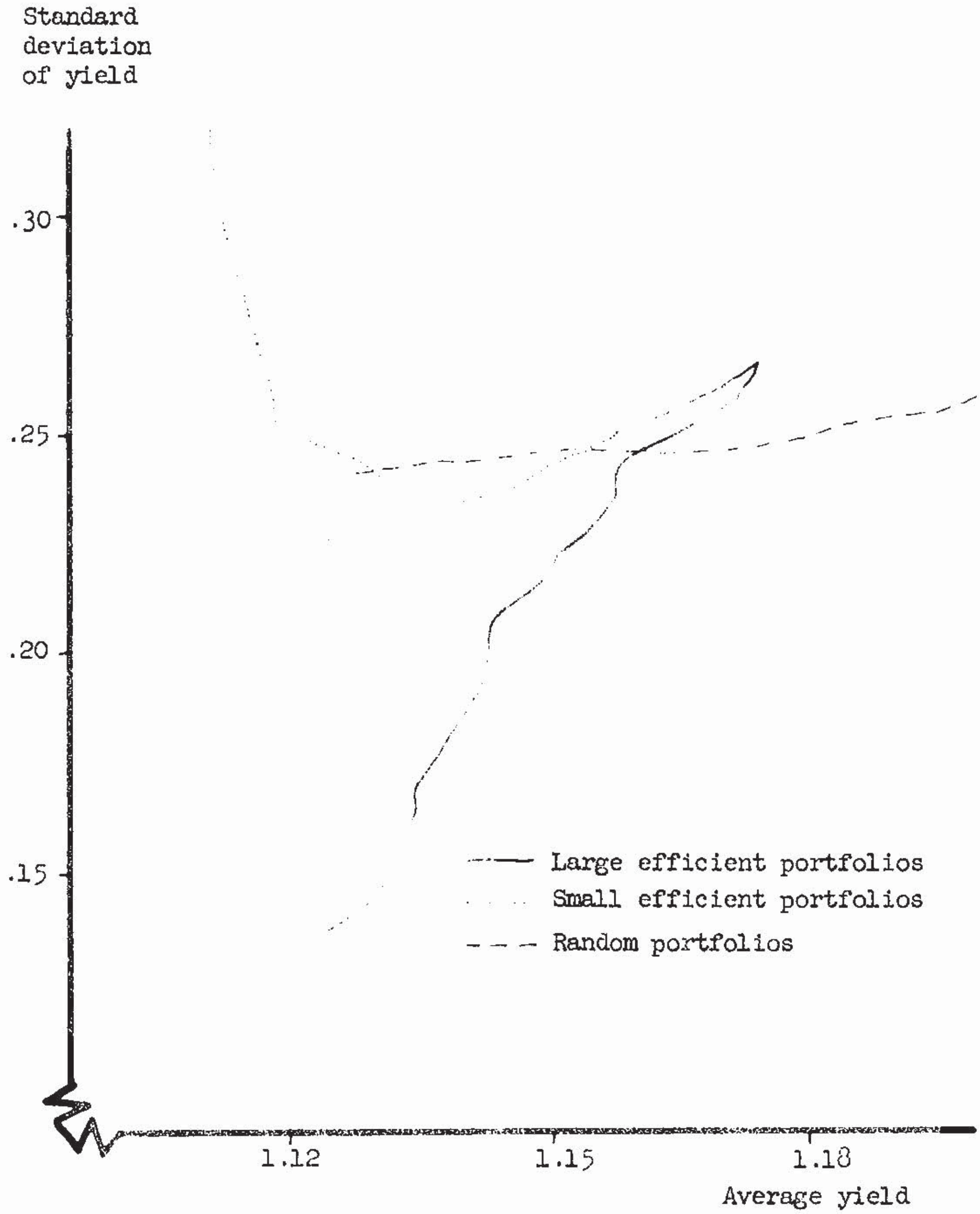

Fig. 17. - Average yield and standard deviation of yield, random and efficient portfolios: 96-security sample

folios. The points have been connected in accordance with the predicted relationships among the various portfolios; the solid segment of the line refers to the portfolios with at least eight securities, the dotted segment shows the performance of the smaller portfolios. The other curve in the figure will be described later.

We have suggested that the diagonal prediction technique appears to offer at least one advantage over random selection of securities: if attention is confined to the larger portfolios, it enables an investor to discriminate between portfolios which are likely to exhibit small average yields (and standard deviations of yield) and those which are likely to exhibit large average yields (and standard deviations of yield). The next question which we seek to answer concerns the relationship between the performance of portfolios chosen by the technique and that of comparable portfolios chosen in some random manner. For convenience, we will refer to portfolios chosen from portfolio analysis with the diagonal technique as "efficient" portfolios, and those chosen by a random technique as "random" portfolios. We wish to determine whether either of these sets of portfolios is dominant; in other words, among portfolios with the same average yield, do the portfolios of one set typically exhibit lower standard deviations of yield?[9]

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[9]The question could as easily have been put in another manner: among portfolios with the same standard deviations of yield, do those from one set typically exhibit higher average yields?

In order to make the comparison, a set of portfolios composed of securities selected with some random technique had to be obtained. Two groups of portfolios were used for this purpose. The first was composed of portfolios containing 50 securities each, on the average. Since there were only 96 securities from which to choose, these portfolios generally had average yields quite close to 1.185, the average yield of the sample during the period. In fact, although 200 such portfolios were selected, too few had average yields below 1.17 to be incorporated in the analysis. In order to obtain random portfolios with lower average yields, a second group of 200 portfolios, containing 14 securities each, on the average, was obtained; these portfolios are comparable in diversification to the larger efficient portfolios. This latter set of random portfolios provided sufficient securities with average yields from 1.14 to 1.17 for the desired comparison. The method used to select the random portfolios is described in the footnote.[10]

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[10] Let N be the number of securities to be included, on the average (N = 50 for the first set, 14 for the second).

1. Consider each security in turn as follows:
    a. Select a random number between 1 and 96;
    b. If the number lies between 1 and N, the security under consideration is to be included in the portfolio; if not, it is to be excluded.
2. For each security selected for inclusion, select a random number between 1 and 100.
3. When all securities have been considered, sum all number selected in (2).
4. For each security selected for inclusion, divide

The additional curve in Fig. 17 shows the median values of standard deviation of yield for all random portfolios which fall within each interval of average yield of width .01. The values are also shown in Table 3.

Table 3

MEDIAN STANDARD DEVIATION OF RANDOM PORTFOLIOS, BY AVERAGE YIELD INTERVAL

Average Yield Interval		Median Standard Deviation
From	To	
1.1300	1.1399	.241
1.1400	1.1499	.242
1.1500	1.1599	.247
1.1600	1.1699	.245
1.1700	1.1799	.247
1.1800	1.1899	.251
1.1900	1.1999	.254
1.2000	1.2099	.260

An alternative method of comparison is shown in Table 4, in which the random portfolios within each average yield interval are divided into three groups: those with smaller standard deviations than those of the efficient portfolios within the interval; those with standard deviations which fall within the range spanned by efficient portfolios within the interval; and those with standard deviations greater than those of the efficient portfolios with in interval.

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the number selected in (2) by the sum computed in (3). The quotient is the amount of the portfolio invested in the security in question.

Table 4

RELATIONSHIP BETWEEN STANDARD DEVIATIONS OF YIELD FOR EFFICIENT PORTFOLIOS AND THOSE OF RANDOM PORTFOLIOS, BY AVERAGE YIELD INTERVAL

Average Yield		Distribution of Random Portfolios*			
From	To	$\sigma_R < \sigma_E$	$\sigma_R = \sigma_E$	$\sigma_R > \sigma_E$	Total
1.130	1.139	0	0	100.0	100.0
1.140	1.149	4.3	35.1	60.6	100.0
1.150	1.159	29.3	19.7	51.0	100.0
1.160	1.169	48.9	16.7	34.4	100.0
1.170	1.179	55.2	3.5	41.3	100.0

*The random portfolios falling within each average yield interval are classified in three groups on the basis of their standard deviation of yield (σ_R). The first group includes random portfolios with smaller standard deviations than those of efficient portfolios within the interval ($\sigma_R < \sigma_E$); the second group includes random portfolios falling within the range spanned by efficient portfolios within the interval ($\sigma_R = \sigma_E$); the third group includes those with standard deviations greater than those of efficient portfolios within the interval ($\sigma_R > \sigma_E$).

Figure 17 and Table 4 both indicate that random and efficient portfolios with equal average yields are likely to have similar standard deviations of yield. This result suggests that the only advantage of the diagonal prediction technique is the one described earlier: it enables an investor to discriminate between portfolios which are likely to exhibit large average yields (and standard deviation of yield) and those which are likely to exhibit small average yields (and standard deviations of yield).

The results of this chapter can be summarized in two hypotheses. First, the diagonal prediction technique is no worse than the historical technique, and since it is cheaper to use it is thus to be preferred. Second, the diagonal technique achieves a fairly accurate ordering of portfolios on the basis of future variability of yield, although the portfolios it selects are not likely to exhibit less variation than will portfolios composed of securities selected at random which have similar average yields. This hypothesis holds only for large efficient portfolios, however (i.e., those with at least eight securities), and only such portfolios should be considered when the diagonal technique is used. Since it is highly desirable to obtain large efficient portfolios, many securities should be analyzed, since larger numbers of securities are likely to result in more diversified efficient portfolios.[11]

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[11]A fair approximation for the cases studied is that the average number of securities in efficient portfolios increases

Obviously, the data presented in this study do not constitute an independent test of these hypotheses, since the hypotheses derive from the data. All that can be said at this point is that the data are not inconsistent with the hypotheses. A true test of the value of objective prediction techniques must await a large-scale study involving many more than 96 securities. Until such a study is performed, it would be premature to reject these techniques; if the hypotheses offered here are valid, they offer considerable advantages over the random selection of securities.

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linearly with the number of securities analyzed. Of course, the effect will be smaller for the less conservative portfolios. At the extreme, the least conservative portfolio will always contain one security, regardless of the number analyzed.

IV. PORTFOLIO ANALYSIS USING SUBJECTIVE PREDICTION TECHNIQUES

A. The Experiment

In order to test the feasibility of using Markowitz's portfolio-analysis technique with subjective predictions of the performance of securities, an experienced investment counselor[1] was asked to provide information from which the parameters of the diagonal model could be computed. The object of this experiment was to determine the extent to which the techniques utilized for summarizing the counselor's beliefs concerning the performance of securities succeeded in communicating his true feelings. A number of aspects of the results provided information on this point: the investment counselor's feelings concerning the validity of his estimates, the extent of internal consistency among them, and the relationship between the counselor's opinions about various portfolios and their characteristics as determined by his explicit predictions of security performance.

To make the communication of the counselor's subjective predictions as simple as possible, agreement was reached on a set of assumptions which formed an intermediate model between the estimates which the counselor provided and the parameters of the diagonal

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[1]The individual who participated in this experiment has been a professional investment counselor for over 13 years and now holds a management position in a leading investment counseling firm.

model. Before reporting the results of the experiment, we will describe both the estimates used and the intermediate model which translated them into the inputs required by the portfolio analysis.

B. Calculation of the Parameters of the Diagonal Model

Figure 18 shows the form used to obtain estimates concerning individual securities.[2] Twenty securities which the counselor considered especially attractive at the time were analyzed. The primary characteristics of the analysis are apparent from an inspection of Fig. 18: the use of a three-year prediction period, the separation of predictions of price changes from those concerning dividends, the choice of the Dow-Jones Industrial Average for a measure of (I),[3] and the use of a probability distribution for estimating future prices.[4] The forms provided the following parameters for each security:

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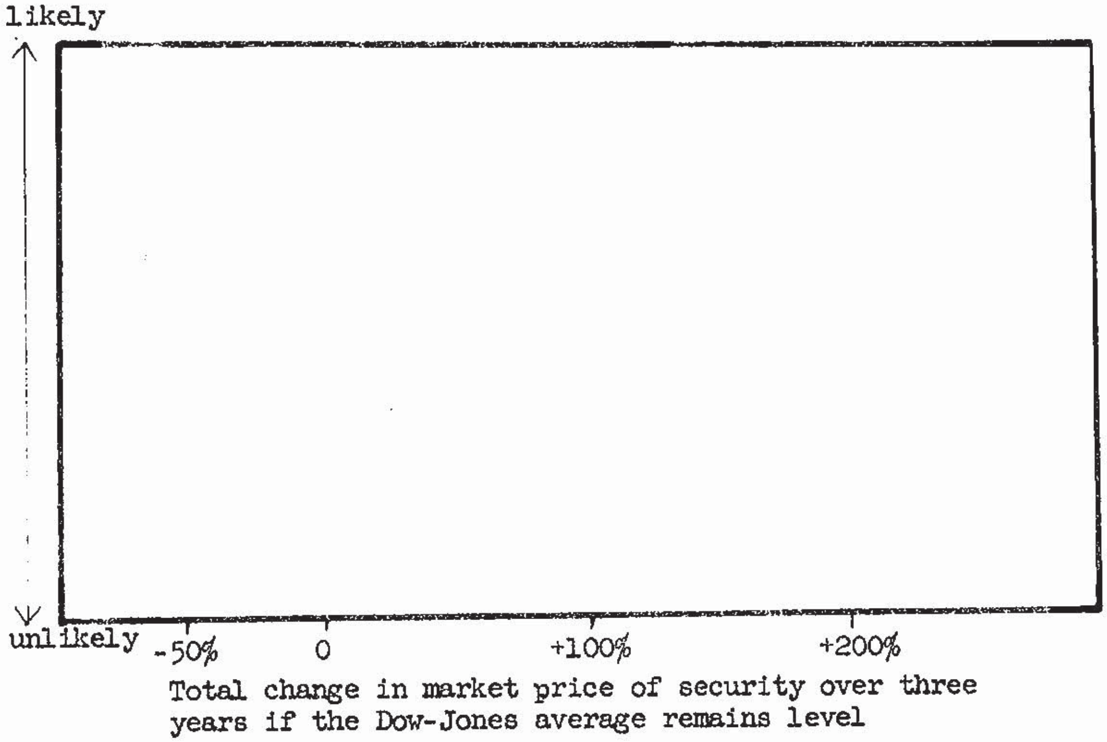
[2]The final question shown in Fig. 18 did not appear on the forms actually used, but was presented later, when its desirability had become apparent.

[3]Although aware of some of the undesirable features of the Dow-Jones Average, the counselor uses it in his analyses because it is prominent in the minds of his clients.

[4]The discussion resulting in the decision to adopt this technique was quite revealing. The author proposed the use of a number of questions such as, "What is the chance of a 10% rise, a 50% rise, etc.?" or, alternatively, "How great a rise might occur one time in twenty?" The investment counselor found such questions unattractive. He finally commented, "It's too bad you won't just let me draw you a graph showing the chances of various prices"!

Security ____________________

Annual dividend yield ______ %



Price of this security if the Dow-Jones average rises to twice its current level:
______ % of the security's current price

Price of this security if the Dow-Jones average falls to 75% of its current level:
______ % of the security's current price

Fig. 18. - Form used for subjective prediction experiment

D_i ≡ the expected value of annual dividend yield;

M_i ≡ the mean value of the probability distribution of the price of the security based on the assumption that the Dow-Jones Average remains level;

U_i ≡ the variance of the distribution;

$P_{1,i}$ ≡ the expected price of the security if the Dow-Jones Average rises to twice its current value;

$P_{2,i}$ ≡ the expected price of the security if the Dow-Jones Average falls to 75% of its current value.

In addition to providing this information for each of the twenty securities, the counselor specified a probability distribution indicating the likelihood of various levels of the Dow-Jones Average at the end of the three-year period. This distribution provided the parameters of the diagonal model which describe the characteristics of I:

A_{n+1} ≡ the mean value of the probability distribution of possible changes in the Dow-Jones Average;

Q_{n+1} ≡ the variance of the distribution.

The price of a security was assumed to be determined in accordance with the relationship:

(9) $$P_i = M_i + R_i \cdot I + z_i$$

where M_i and (I) are as defined above, R_i indicates the response of the price of a security to changes in (I), and z_i is a random variable with a mean of zero and a variance of U_i. R_i is the only parameter of this equation which has not yet been defined.

Figure 19 illustrates the calculation of R_i; the horizontal

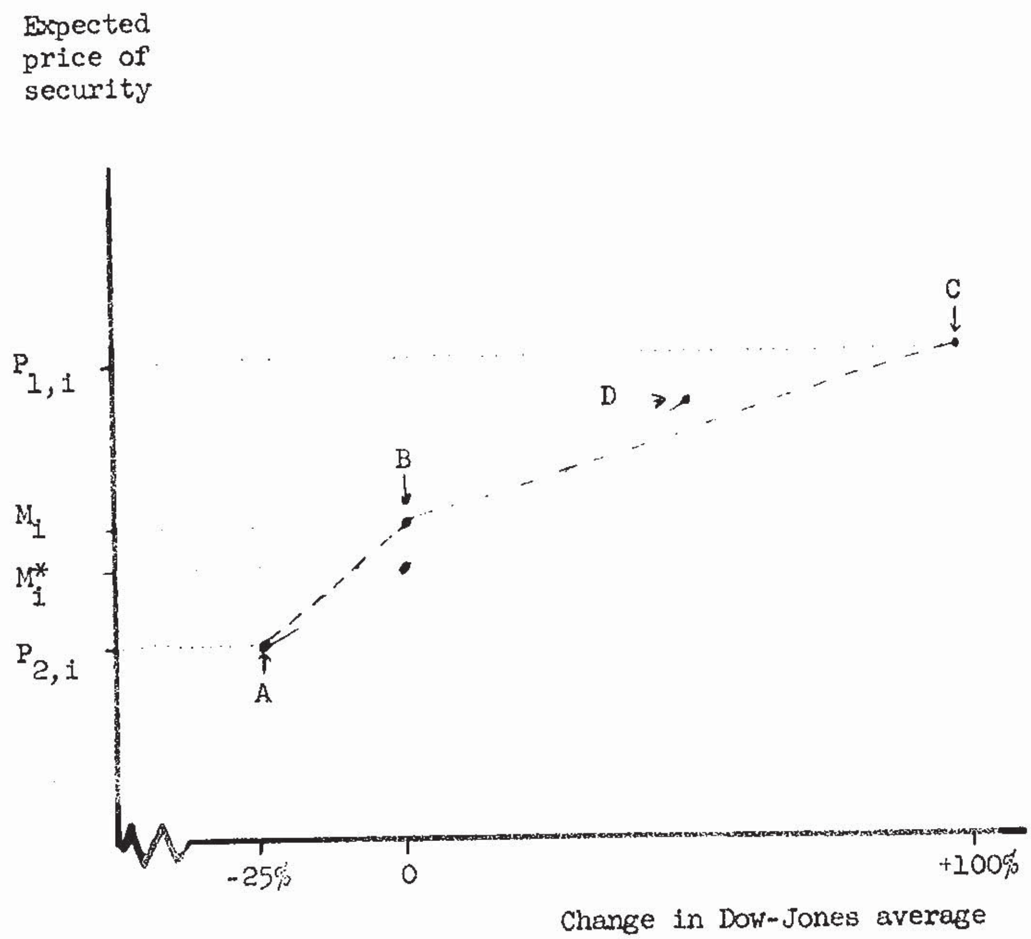


Fig. 19. - Calculation of parameters in subjective prediction experiment

axis indicates alternative changes in the Dow-Jones Average, the vertical axis indicates the associated expected price of a security. M_i, $P_{1,i}$ and $P_{2,i}$ are the quantities previously defined. The curve ABC shows the typical relationship among these three values for the twenty securities analyzed, indicating that the counselor's beliefs do not completely conform to the assumption of the diagonal model that this relationship is linear.

In order to select the important range of changes in the Dow-Jones Average, the appropriate distribution was examined: it had an expected value of + 50% and indicated a very slight chance of a change below - 25%. Since the object of portfolio analysis is to minimize losses associated with outcomes below expected values, this is the range of primary importance. The estimate of R_i was obtained by averaging the slopes of the lines AB and BC; and the value of M_i replaced with M_i^* --calculated so that $P_{2,i}$ remains the expected price when the change in the Dow-Jones Average is - 25%. The line AD in Fig. 19 illustrates the relationship used for the final calculations.

The total yield of a security includes both price changes and dividends; since dividends are generally paid quarterly we convert each estimate of annual yield into a comparable quarterly figure, d_i ($\equiv D_i/4$) and, for convenience, assume that this yield relates to the price at the close of each quarter. Then, if P_1 is the price of a security at the beginning of the first quarter and P_2 its price at the end of the quarter, dividends paid during the period are dP_2 and the total yield is:

$$Y_1 = \frac{P_2 + dP_2}{P_1} = (1 + d)\,\frac{P_2}{P_1}$$

Similarly, the yield during the second quarter is:

$$Y_2 = \frac{P_3 + dP_3}{P_2} = (1 + d)\,\frac{P_3}{P_2}$$

The yield of the security over the twelve quarters of the three-year prediction period is merely the product of the quarterly yield figures:

$$Y = (1 + d)\,\frac{P_2}{P_1} \cdot (1 + d)\,\frac{P_3}{P_2} \;\; . \; . \; . \;\; (1 + d)\,\frac{P_{12}}{P_{11}}$$

$$= (1 + d)^{12}\,\frac{P_{12}}{P_1}$$

But (P_{12}/P_1) is merely the P of our previous equations. Thus:

$$Y_i = (1 + d)^{12} \cdot P_i$$

(10) $$Y_i = (1 + d)^{12}\left[M_i^* + R_i \cdot I + z_i\right]$$

The relationship between the parameters of the diagonal model and the estimates provided by the investment counselor is implicit in Eq. (10):

$$A_i = (1 + d)^{12} \cdot M_i^*$$
$$B_i = (1 + d)^{12} \cdot R_i$$
$$Q_i = (1 + d)^{24} \cdot U_i$$

C. Results of the Analysis

Figure 20 shows the results of a portfolio analysis using the parameters obtained from the investment counselor's beliefs. The solid line indicates combinations of predicted average yield and standard deviation of yield for efficient portfolios. (In reading the figure, remember that all values shown refer to performance over a three-year period.) The predicted performance of each of the twenty securities is also shown; the dotted line and the boundary of efficient combinations together enclose the space of attainable combinations of average yield and standard deviation of yield. Any portfolio composed of one or more of the twenty securities would have a predicted performance shown by some point within the space.

As part of the experiment, the investment counselor was asked to specify three portfolios of different conservatism, which he would recommend to clients. The predicted performance of these portfolios was then calculated, using the predictions of individual security performance obtained previously. The three portfolios are shown as points 1, 2, and 3 in Fig. 21, which is a larger-scale illustration of a portion of Fig. 20. The predicted performance of the three portfolios is consistent with the investment counselor's descriptions: Portfolio 1 was described as the most conservative, Portfolio 2 as fairly conservative, and Portfolio 3 as least conservative.

The three portfolios selected by the investment counselor will be compared with two alternative groups of portfolios:

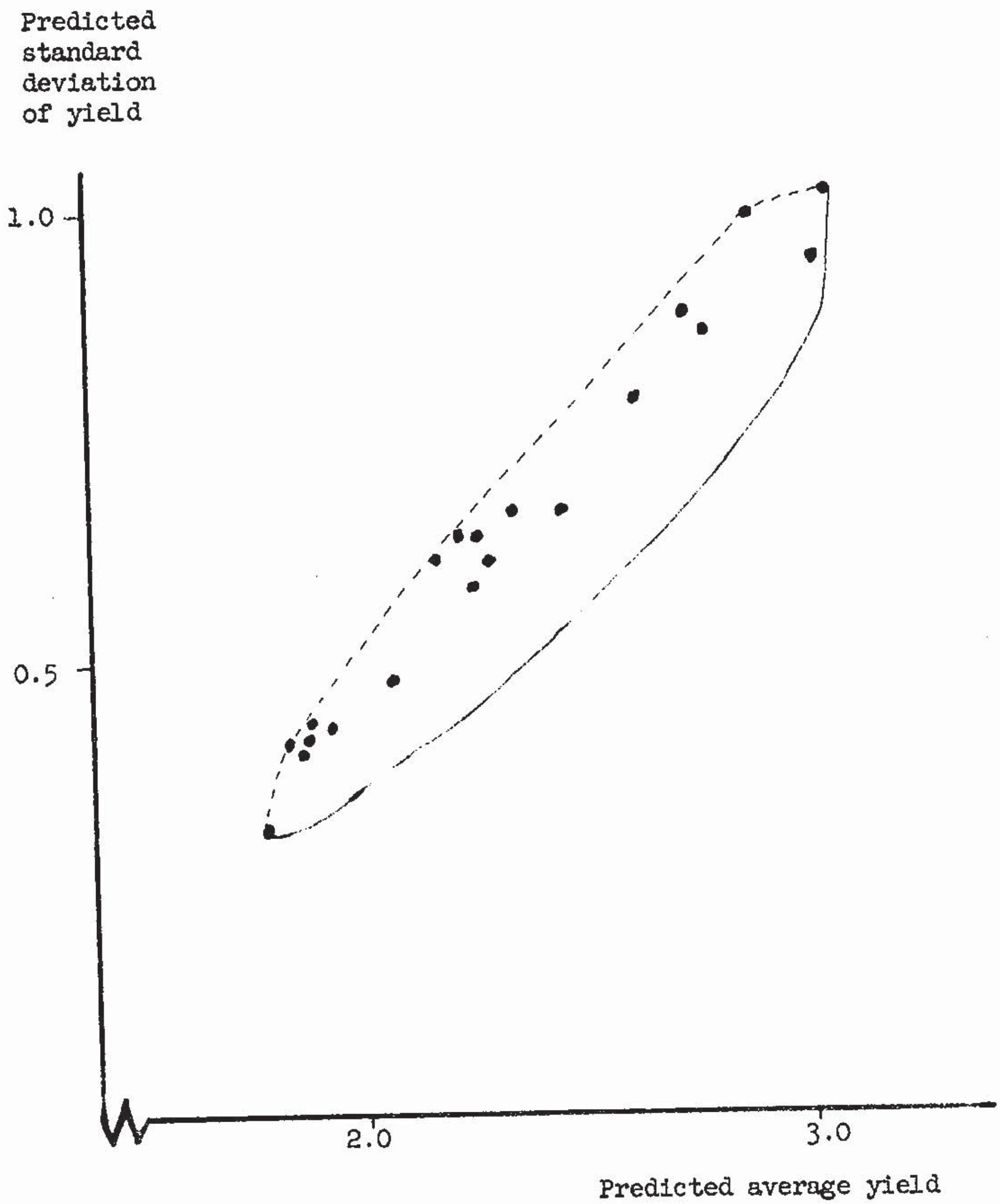


Fig. 20. - Predicted values of average yield and standard deviation of yield for efficient portfolios: subjective prediction experiment

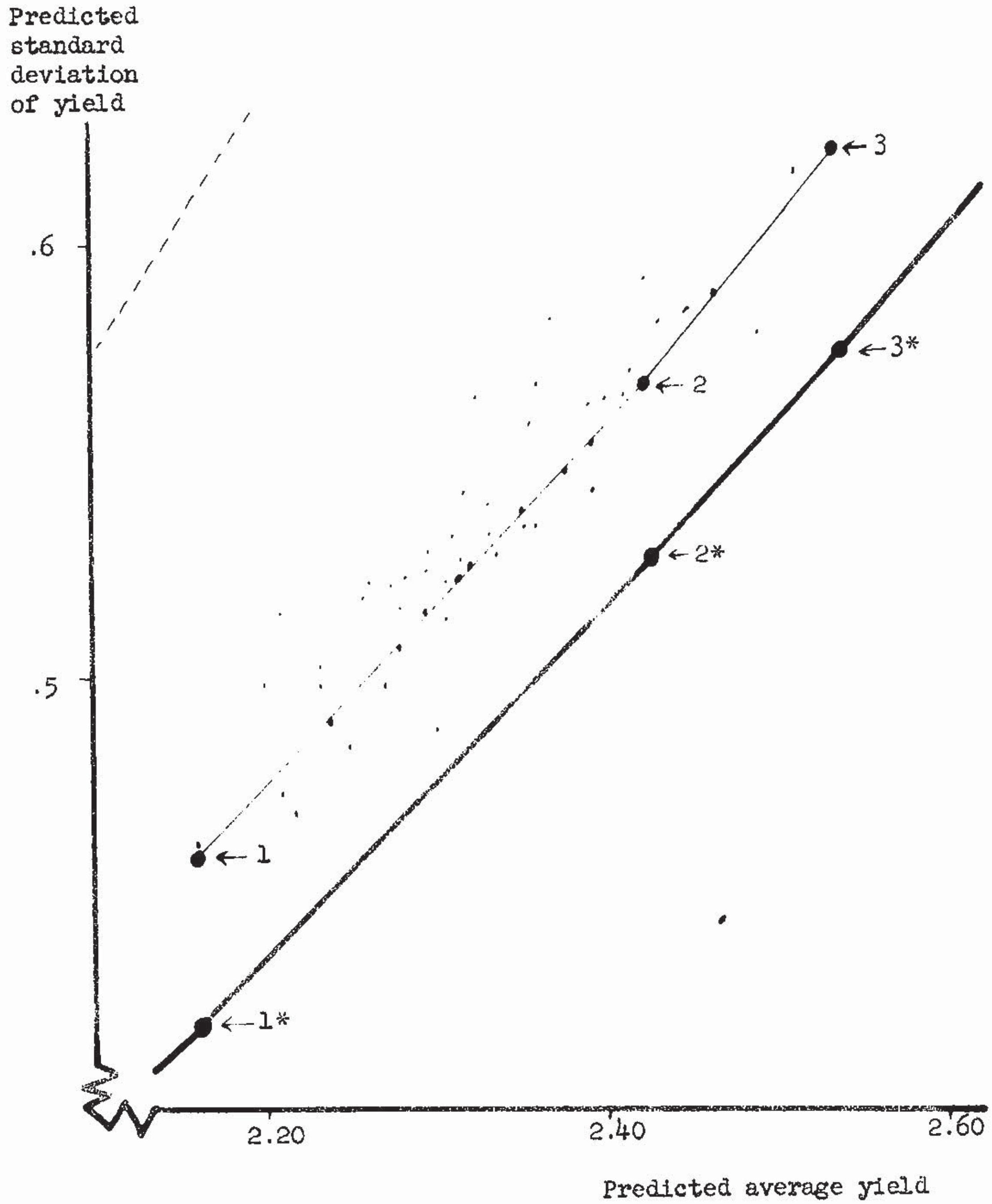


Fig. 21. - Predicted values of average yield and standard deviation of yield, random and efficient portfolios: subjective prediction experiment

random and efficient. We first consider portfolios resulting from the random selection of securities. Figure 21 shows the predicted performance of 50 such portfolios, each containing about 15 securities.[5] For purposes of comparison, points 1, 2, and 3 have been connected with straight lines; the resulting curve can be used as an estimate of the predicted performance of all portfolios which the counselor might choose in this range of predicted average yield.[6] Two aspects of the relationship between this curve and the points representing the random portfolios are of interest. First, none of the random portfolios had a predicted average yield or standard deviation of yield which lay outside the range covered by Portfolios 1 and 3; the counselor's selections thus bracket the range of results attainable with fairly sizeable portfolios. Second, and more important, the counselor's portfolios are generally more efficient than those composed of a fairly large number of securities selected randomly: of the 50 random portfolios, 11 lie below the curve, 9 are on it, and 30 lie above it.

The second comparison is between the investment counselor's

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[5] The selection technique was that described in Chapter III with N = 15. The only other change occurred in Step (1a), in which a number between 1 and 20 was chosen, rather than one between 1 and 96.

[6] The straight line between points 1 and 2, for example, is an underestimate of the performance of all combinations of Portfolios 1 and 2. The true curve lies slightly below and to the right of this line, but the difference is slight.

selections and efficient portfolios. For this purpose, three portfolios (1*, 2*, and 3*) were selected from the set of efficient portfolios. Each has the same predicted average yield as one of the investment counselor's portfolios, as shown in Fig. 21. The composition of all six portfolios is given in Table 5.

The differences between the investment counselor's portfolios and comparable efficient portfolios are striking. According to the parameters obtained in the experiment, the counselor's selections are much less efficient--a result related to the fact that the efficient portfolios are much less diversified than those chosen by the counselor. This lack of diversification was the result of a rather large divergence between the parameters given three securities and those assigned the others;[7] the predictions for these three securities made them so attractive that their inclusion as major elements of a portfolio more than offset the resulting lack of diversification and associated risk.

Two interpretations of these results can be offered. The first would accept the validity of the parameters used in the portfolio analysis and thus its conclusions, with the implication that the counselor had failed to realize that the advantages of the three exceptional securities were so great that he should concentrate his holdings heavily on them. The second interpretation would question

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[7]The particular aspects of the parameters which made these securities so desirable are described in the next chapter.

Table 5

PERCENTAGES OF PORTFOLIO INVESTED IN SECURITIES

	Portfolio					
Security	1	1*	2	2*	3	3*
1	4.5		9.0	2.8	9.0	4.1
2	10.4	13.9	5.2	14.1	5.2	13.0
3	3.9		3.8	0.1	3.9	0.1
4			4.5	1.0	9.1	2.4
5	4.6		4.5		4.6	
6	7.3	13.5	7.3	18.6	3.7	19.8
7	7.4		7.4		5.6	
8	3.0		6.0		6.0	
9	3.8		7.5	3.8	7.5	6.1
10		15.2	7.2	23.6	9.0	26.7
11	9.2		4.6		4.6	
12	3.3	14.8	6.5	22.7	6.5	25.2
13	8.6		6.4		4.3	
14	6.5	1.1	3.2			
15	4.6		4.6		9.2	
16	4.6	5.3	3.1		3.1	
17	6.7	35.2	3.3	13.3		2.6
18	5.7	1.0				
19	5.9					
20			5.9		8.7	
Average Yield	2.157	2.157	2.426	2.426	2.537	2.537
Standard Deviation of Yield	0.459	0.417	0.568	0.527	0.622	0.575

the validity of the parameters, especially those relating to the three exceptional securities. Discussions with the investment counselor after the portfolio analysis was completed indicated that the latter interpretation is to be preferred. While agreeing that the three securities were exceptionally desirable, he felt quite strongly that he would never concentrate a portfolio in those securities as heavily as did the portfolio analysis. He was more than willing to admit that his estimates of the performance of those securities had apparently overstated his true feelings.

The results of this experiment emphasize the rigor of the portfolio-analysis technique. The Markowitz approach takes into account all differences among parameters, some of which may be much smaller than the probable error in the estimates; one consequence is that relatively few securities enter efficient portfolios, a result which is likely to prove unacceptable to most investors. While the other relationships investigated in this experiment suggest that the parameters obtained from the investment counselor contain a great deal of information concerning his beliefs, it is evident that they fail to estimate precisely the absolute magnitude of differences among the parameters. If this is typically the only error, however, the Markowitz portfolio-analysis technique may still make a contribution in the process of investment selection using subjective prediction techniques. If a large enough number of securities were analyzed, the resulting efficient portfolios would probably be fairly diversified. While it is unlikely that these portfolios would prove much more efficient than those selected by the analyst himself, use of the Markowitz technique could facilitate an increased division of labor--allowing much of the task of portfolio selection to be delegated to subordinates. The investment counselor who participated in this experiment indicated that this use of the technique would be of some value in his firm. Although the results are likely to be modest, it appears that an attempt to apply the Markowitz technique to subjective predictions of the performance of a large number of securities deserves serious consideration.

V. A POSITIVE THEORY OF SECURITY MARKET BEHAVIOR

A. The Theory

In this chapter we turn from normative applications of portfolio analysis to an attempt to construct a positive theory of security market behavior. The basic assumptions of this theory are: (1) investors act as if they were applying the Markowitz portfolio-analysis technique to probability beliefs about securities, and (2) their beliefs are expressed in terms of the diagonal model.

A number of relationships must be established before the market equilibrium conditions which result from these assumptions can be described. The first concerns the effect of portfolio size on variance. This can best be illustrated by considering alternative portfolios, each of which contains N securities, with (1/N) of the value of the portfolio invested in each security.[1] The variance of such a portfolio would be:

$$V = (X_{n+1})^2 \cdot V_I + \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \cdot Q_i$$

$$= (X_{n+1})^2 \cdot V_I + \frac{\sum_{i=1}^{N} Q_i}{N^2}$$

[1]The results we will obtain can be shown to hold in the general case, where the amounts invested in different securities are unequal.

where:

$$X_{n+1} = \sum_{i=1}^{N} \left(\frac{1}{N}\right) B_i = \frac{\sum_{i=1}^{N} B_i}{N}$$

Letting $\bar{B}$ be the average value of B_i and Q the average value of Q_i for the N securities, the formula becomes:

$$V = (\bar{B})^2 \cdot V_I + \frac{\bar{Q}}{N} \quad .$$

This formulation indicates that the larger the number of securities in a portfolio, the less significant is the portion of variance due to the Q_i parameters. With sufficiently large portfolios, this latter type of risk can be virtually disregarded since the variance of the portfolio will be due almost entirely to the B_i parameters. This is intuitively plausible. The parameter Q_i reflects a risk unique to a security which is in no way correlated with the performance of any other security; diversification can reduce this type of risk to negligible proportions. On the other hand, the parameter B_i refers to risk which is related to the performance of other securities; it is an implicit statement of the covariance between the yield of the security and that of all others. This type of risk cannot be offset by diversification and remains important regardless of the size of a portfolio.

Figure 22 illustrates the typical relationship between V and N, using median values of the 96-security sample during the period 1940-51 for $\bar{B}$ (=0.9) and $\bar{Q}$ (=0.05); the performance of the

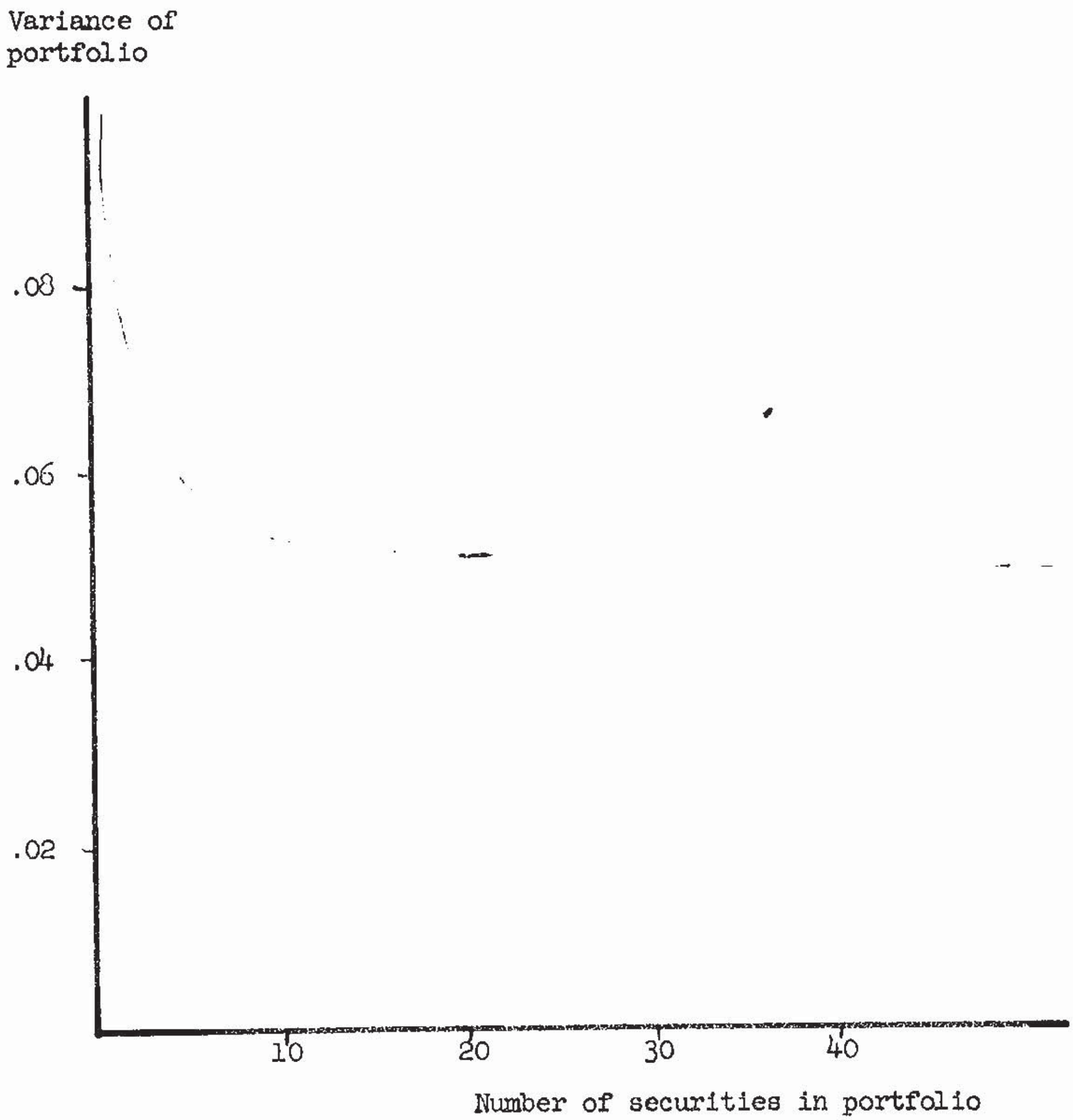

Fig. 22. - Predicted relationship between the number of securities in a portfolio and its variance

96-security index during the period was used as an estimate of V_I (=0.06). The curve indicates the relatively small number of securities required to reduce variance to nearly its minimum value. Since so few securities are typically required to reduce the variance of a portfolio to the amount due to the $\bar{B}$ parameter, and since so many securities are available to investors, we would expect to find that investors place primary emphasis on B_i when assessing the risk of a security, devoting much less attention to the Q_i parameter.

Consider next the effects of changes in the price of a security on the parameters B_i and E_i. Assume that investment in one share of a particular security will result in a value of $\$Z_i$ at the end of one year, where $\$Z_i$ includes both the price of the security at the time and all dividends paid during the year. Assume further that the value of Z_i will be determined by the formula:

$$Z_i = F_i + G_i \cdot I + h_i \tag{11}$$

where (I) represents the deviation of the market index from its expected value (so that $E(I) = 0$), F_i and G_i are parameters, and h_i is a random variable with a zero mean. In this discussion we will assume that the initial price of the security will not affect the determination of Z_i. A number of alternative assumptions could be presented, but since they typically affect only the adjustment process, yielding the same equilibrium condition, we will limit our discussion to this case.

If Z_i is the total dollar value at the end of the year resulting from the purchase of one share of a security, and P_i is the price paid for that share at the beginning of the year, then, by our definition, the yield is:

$$Y_i = \frac{Z_i}{P_i}$$

Substituting Eq. (11) for Z_i:

$$Y_i = \frac{(F_i + G_i \cdot I + h_i)}{P_i}$$

Or:

$$Y_i = \frac{F_i}{P_i} + \frac{G_i}{P_i} \cdot I + \frac{h_i}{P_i}$$

Compare this formulation with the basic equation of the diagonal model:

$$Y_i = A_i + B_i \cdot I + w_i$$

Obviously:

$$A_i = \frac{F_i}{P_i}$$

$$B_i = \frac{G_i}{P_i}$$

$$w_i = \frac{h_i}{P_i}$$

We have defined (I) so that its expected value is zero. Thus:

$$E_i = A_i = \frac{F_i}{P_i}$$

These equations indicate that a change in the price of a security will affect both E_i, its expected yield, and B_i, the relationship between its yield and the market index. E_i and B_i will both move inversely with the price of the security; note, however, that their ratio is independent of the price:

$$\frac{B_i}{E_i} = \frac{\frac{G_i}{P_i}}{\frac{F_i}{P_i}} = \frac{G_i}{F_i}$$

These relationships are illustrated in Fig. 23. As the price of security R rises, the point showing its characteristics will move to the left along line 1; as its price falls, the point will move to the right along the same line.

Another relationship of importance in this analysis concerns the effects of combining portfolios. Since both B and E are linear functions of the quantities of the various securities included in a portfolio, the values of E and B pertaining to a portfolio formed by investing partly in one portfolio and partly in another will lie on the line connecting the values of E and B of the original portfolios. Thus, in Fig. 23, point T could represent an investment divided between portfolios W and U.

We have considered the characteristics of securities; we now

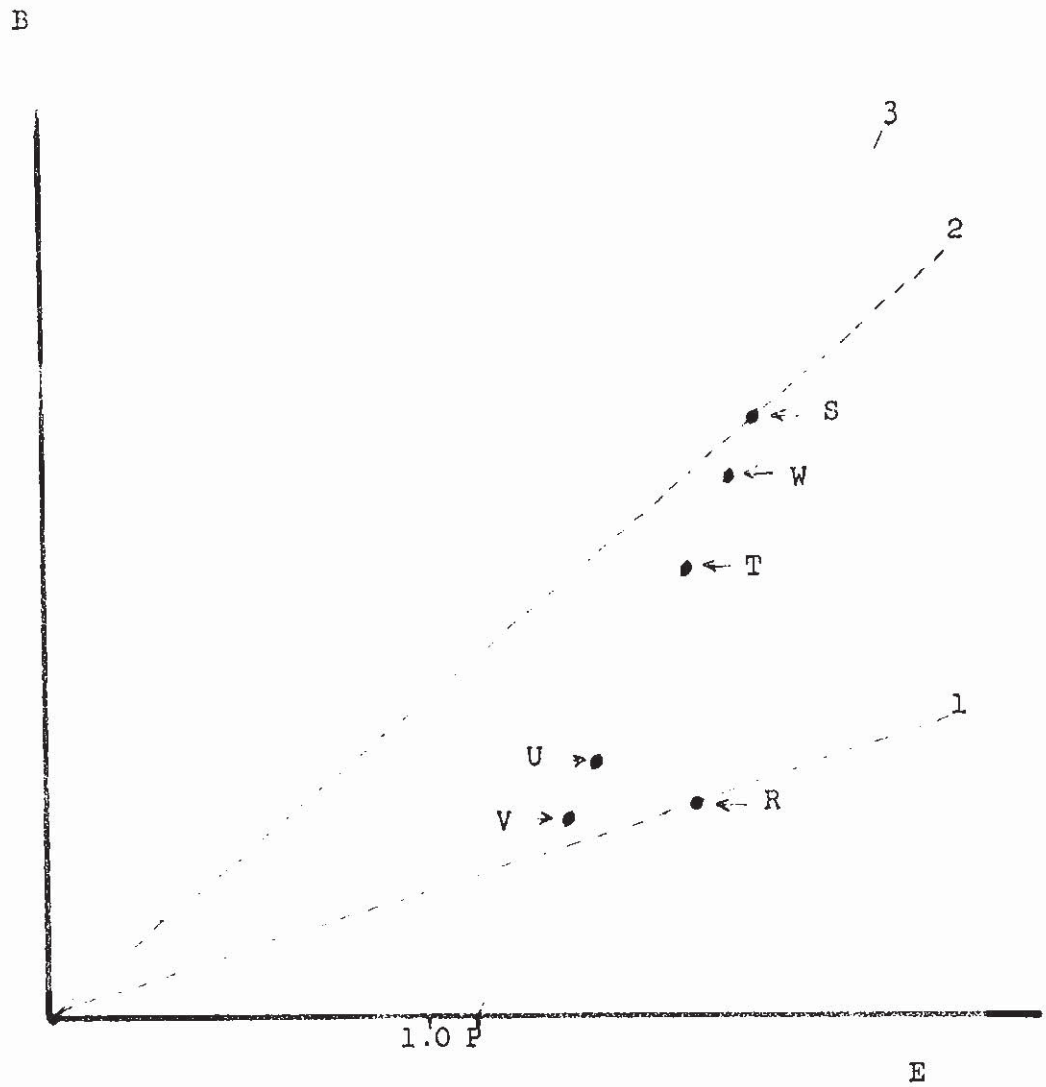

Fig. 23. - Disequilibrium in the security market

must consider one very important alternative investment: lending money at the "pure" interest rate. This alternative has an expected return equal to the pure interest rate ($E_i = 1 + r$), but there is no associated risk; thus B_i and Q_i are both zero. Since this can be considered merely another portfolio, combinations of lending money and regular portfolios will have predicted values of B and E which lie on the line connecting the two investments. Thus point U in Fig. 23 could be obtained by investing some portion of a fund in portfolio T and lending the remainder at the pure interest rate, shown by point P.

One additional characteristic of the pure-interest-rate alternative is important. It is possible to "disinvest" in a security (or portfolio) by borrowing rather than lending money at this rate;[2] thus it would be possible to obtain point U in Fig. 23 by borrowing money to purchase portfolio V. In general, by borrowing or lending, it is possible to obtain any combination of B and E falling on a line passing through point P in Fig. 23 and any point representing some portfolio or security.

The equilibrium process can now be shown. In Fig. 23 the points S, U, and V are a few of many securities which lie along line 3, which originates at point P. We have shown that the values of E and B associated with any portfolio composed of these securities will

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[2]We assume equal borrowing and lending rates. The implications of differential rates are of considerable interest but will not be discussed here.

lie along line 3; of all such portfolios, one will have the smallest variance due to the Q_i parameters. Let this portfolio be represented by point W. Since there will be many securities along line 3, the variance of this portfolio which is caused by the Q_i parameters will be so close to zero that it can be disregarded.

Next consider the effects of borrowing and lending at the pure interest rate. Any combination of B and E along the line PW can be obtained by some combination of portfolio W and lending money. Since the variance due to the Q parameter is zero for both components of such an investment, its entire variance will be due to the B parameter. On the other hand, any combination of B and E along line 3 above point W can be obtained by borrowing money to purchase additional amounts of portfolio W; again, the variance will be due entirely to the B parameter.

Now assume that some security, such as R, lies to the right of line 3. Compare it with the combination of W and P represented by point T: both have the same expected yield but R has a smaller B-parameter and a greater Q-parameter. Under these conditions it would be profitable for investors to include security R in their portfolios; the proper amount would depend on the relationship between the parameters: the larger the difference in the B parameters relative to the Q parameter of R, the larger the proportion of the portfolio which should be devoted to R.[3]

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[3]Assume that portfolio T has a B-parameter of $\bar{B}$ and no variance due to Q_i parameters. Then if B_R and Q_R are the parameters of

What would be the market response to a security which (temporarily) had characteristics similar to those described for

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security R, and X_R is the amount of a portfolio invested in R, the variance due to Q_R will be:

$$V_Q = (X_R)^2 \cdot Q_R$$

$$\frac{d(V_Q)}{d(X_R)} = 2Q_R \cdot X_R$$

Letting ΔB be the difference $(\bar{B}-B_R)$, the B parameter of the new portfolio will be:

$$B = \bar{B} - X_R \cdot \Delta B$$

and the variance due to this parameter will be given by:

$$V_B = (\bar{B} - X_R \cdot \Delta B)^2 \cdot V_I$$

$$\frac{d(V_B)}{d(X_R)} = 2V_I(\Delta B)\left[X_R \Delta B - \bar{B}\right]$$

The optimum amount of security R can be found by equating the increase in variance due to Q_R with the absolute value of the decrease due to B_R as X_R is increased:

$$2Q_R X_R = -\left(2V_I\ (\Delta B)\left[X_R \cdot \Delta B - \bar{B}\right]\right)$$

The optimum is thus:

$$X_R = \frac{V_I \bar{B} \Delta B}{(\Delta B)^2 \cdot V_I + Q_R}$$

Rewriting:

$$X_R = \frac{V_I \bar{B}}{\Delta B \cdot V_I + \frac{Q_R}{\Delta B}}$$

This formulation shows that, ceteris paribus, the higher is Q_R relative to ΔB, the larger the amount of the portfolio which should be invested in security R, thus proving the statement in the text.

security R? People would desire to buy more of this security and less of those securities with higher values of B, causing an increase in the price of R and a decrease in the prices of all securities along line 3. In Fig. 23 point R would move to the left along line 1, while points S, U, and V would move to the right along their respective lines from the origin, establishing a new line from point P lying to the right of line 3. If point R remained below this new line, similar incentives would exist for increased purchase of R and decreased purchases of S, U, and V, resulting in further movements of the prices of the securities.

Figure 24 indicates the equilibrium condition in the security market. All securities will lie along one straight line which cuts the E-axis at the pure interest rate. When a security temporarily moves to the right of the line, forces are set in motion which move the line to the right and the point representing the security to the left. On the other hand, when a security temporarily moves above the line, its price will fall, moving its point to the right, while the prices of securities on the line will rise, moving the line to the left. Only when conditions are as pictured in Fig. 24 will equilibrium be achieved.

Figure 24 indicates the combinations of B and E for individual securities which will occur when the market is in equilibrium; Fig. 25 illustrates the relationship between standard deviation of yield and average yield for portfolios under these conditions. We have shown that any combination of B and E on line PP' in Fig. 24

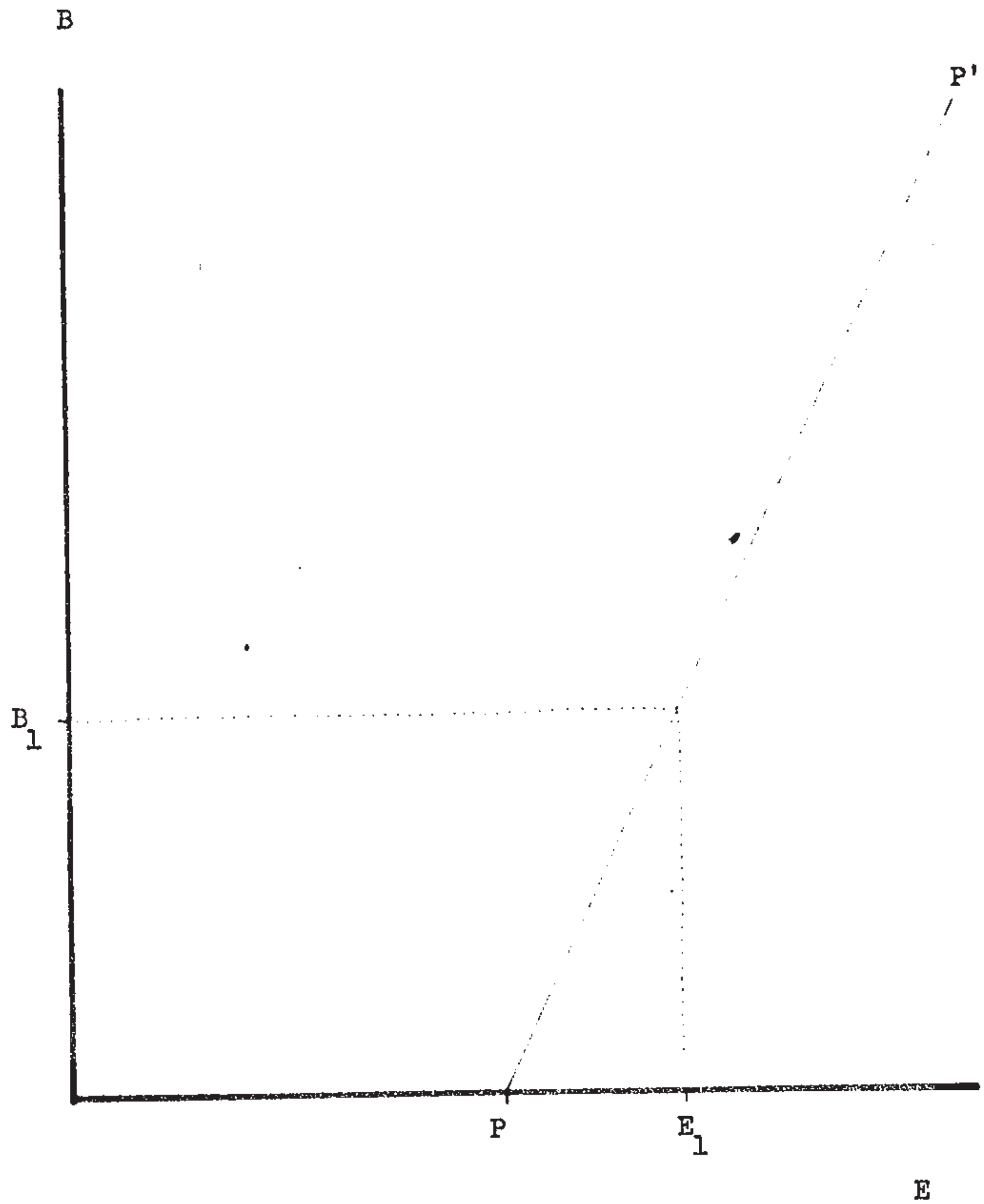

Fig. 24. - Equilibrium in the security market

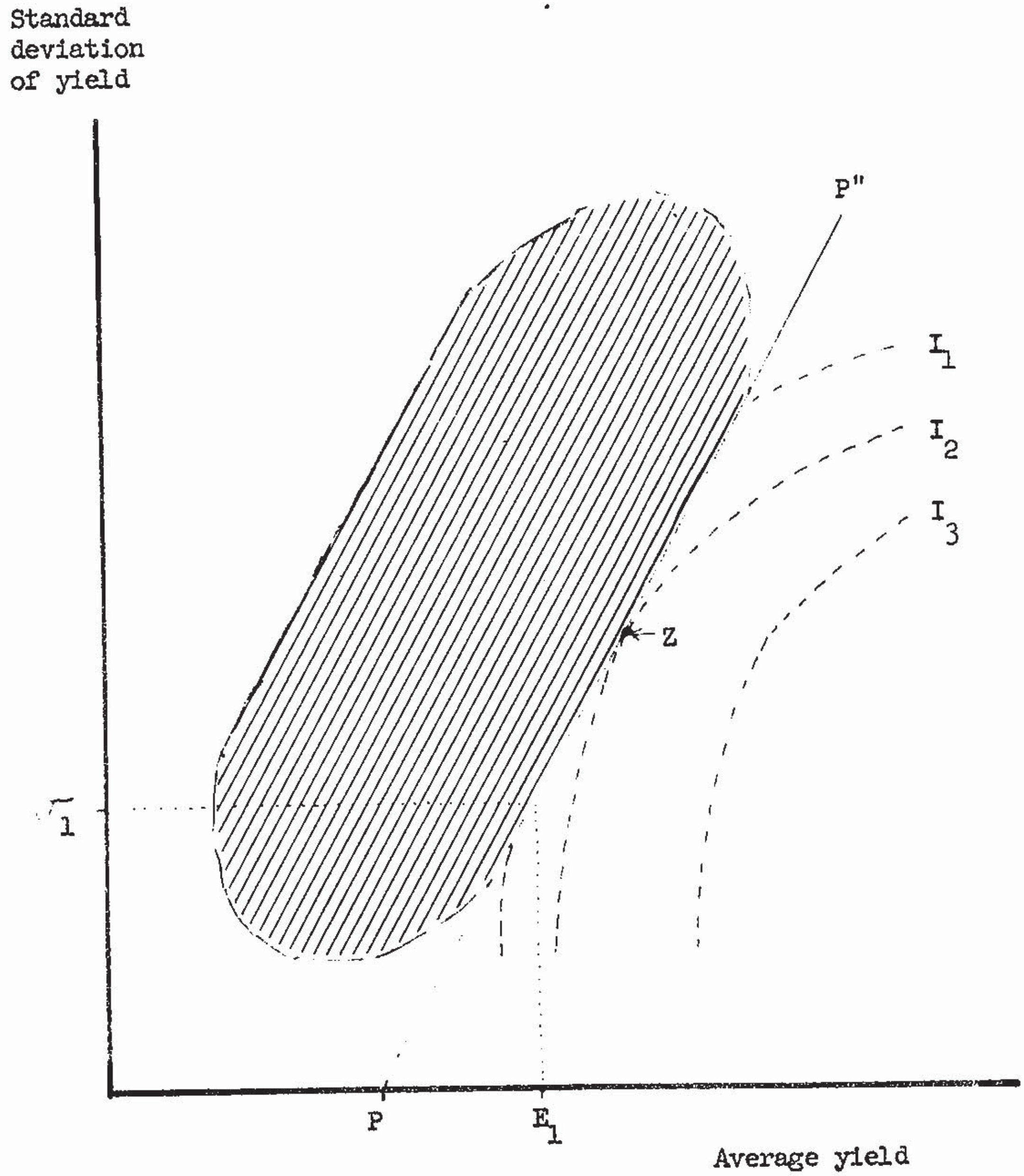

Fig. 25. - Equilibrium relationship between average yield and standard deviation of yield for portfolios

can be attained by some portfolio with zero variance due to Q. For example, there is a portfolio with an expected yield of E_1, a market response of B_1, and a variance due to non-market factors of zero. The total variance of such a portfolio is $(B_1)^2 V_I$ and its standard deviation $B_1 \sigma_I = \sigma_1$. For all portfolios with zero variance due to Q, B is a linear function of E -- line PP' in Fig. 24 -- and σ is a linear function of B; thus σ will be a linear function of E -- line PP" in Fig. 25. Note that this line also cuts the E-axis at the pure interest rate.[4]

We have shown that all portfolios for which the variance due to Q is zero will lie along line PP" in Fig. 25. But many possible portfolios (in particular, one-security portfolios) do not have zero variance due to Q; they will lie above the line, since they have as much variance due to B as do comparable efficient portfolios plus an additional variance due to Q. Such portfolios will lie within a space similar to that shown in Fig. 25. When the market has reached equilibrium, no quadratic program is required to distinguish between inefficient and efficient portfolios; the set of efficient portfolios is merely the set of diversified portfolios.

The investor who assumes that the security market is in equilibrium approaches the problem of portfolio selection in much

---

[4]Since B and σ for securities are likely to be correlated, the relationship between σ and E for single securities may also be a straight line intersecting the E-axis at the pure interest rate. This was the case for the estimates provided by the investment counselor, as can be seen in Fig. 20.

the same manner as the consumer approaches the problem of allocating his income between two goods. Having determined the attainable combinations of average yield and standard deviation of yield (line PP" in Fig. 25, the investor selects the combination which will maximize his utility. In Fig. 25 this combination is shown by point Z, where indifference curve I_2 touches line PP". In reading this figure, remember that since higher values of σ are less desirable than lower values, higher levels of utility lie to the right and downward.[5] An alternative interpretation of the portfolio-selection problem characterizes each investor as adjusting his holdings until his marginal rate of substitution of risk for earnings equals the common market rate. Of course the market rate is itself determined by the preferences of investors.

B. Arbitrage in the Security Market

We have described a process through which equilibrium in the security market can be attained. Instrumental in this process are arbitragers -- investors who attempt to discover securities which are temporarily undervalued or overvalued, and who, through their purchases and sales of such securities, bring about the price changes which restore equilibrium. The arbitrager does not believe

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[5]This discussion assumes, of course, that investors wish to avert risk. For an excellent discussion of this problem (in a somewhat different context) see James Tobin, "Liquidity Preference as Behavior Towards Risk," The Review of Economics and Statistics, XXV (February, 1958), 65-86. Tobin derives a linear relationship between σ and E using a highly simplified model in which the alternative investments are cash and consols.

that all securities lie along a line similar to PP' in Fig. 24; he believes that some overvalued securities lie above such a line and that some undervalued securities lie below it. He rejects the former from consideration entirely, leaving only the undervalued securities plus those which he considers to be priced correctly in the set of securities from which efficient portfolios will be constructed.

Figure 26 serves as a convenient illustration of this process; it shows the values of B and E for the twenty securities selected by the investment counselor in the experiment described in the previous chapter. Notice that all but three of the securities (numbers 6, 10, and 12) lie along a line which passes through the pure interest rate;[6] the three exceptional securities are the ones which entered the set of efficient portfolios in such large amounts. Figure 26 indicates the reason. As indicated previously, when securities lie below the line, it will prove profitable to include them in portfolios, the amount depending on the magnitude of their Q-parameters relative to the distance from the line. All three of these securities had high Q-parameters but lay so far below the line (especially number 10) that the portfolio-analysis technique relied heavily on them in forming the set of efficient portfolios.

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[6]The line was fitted free-hand. It passes through 1.12 at its intersection with the E-axis. This corresponds to an annual interest rate of 3.9% since all values refer to a three-year period.

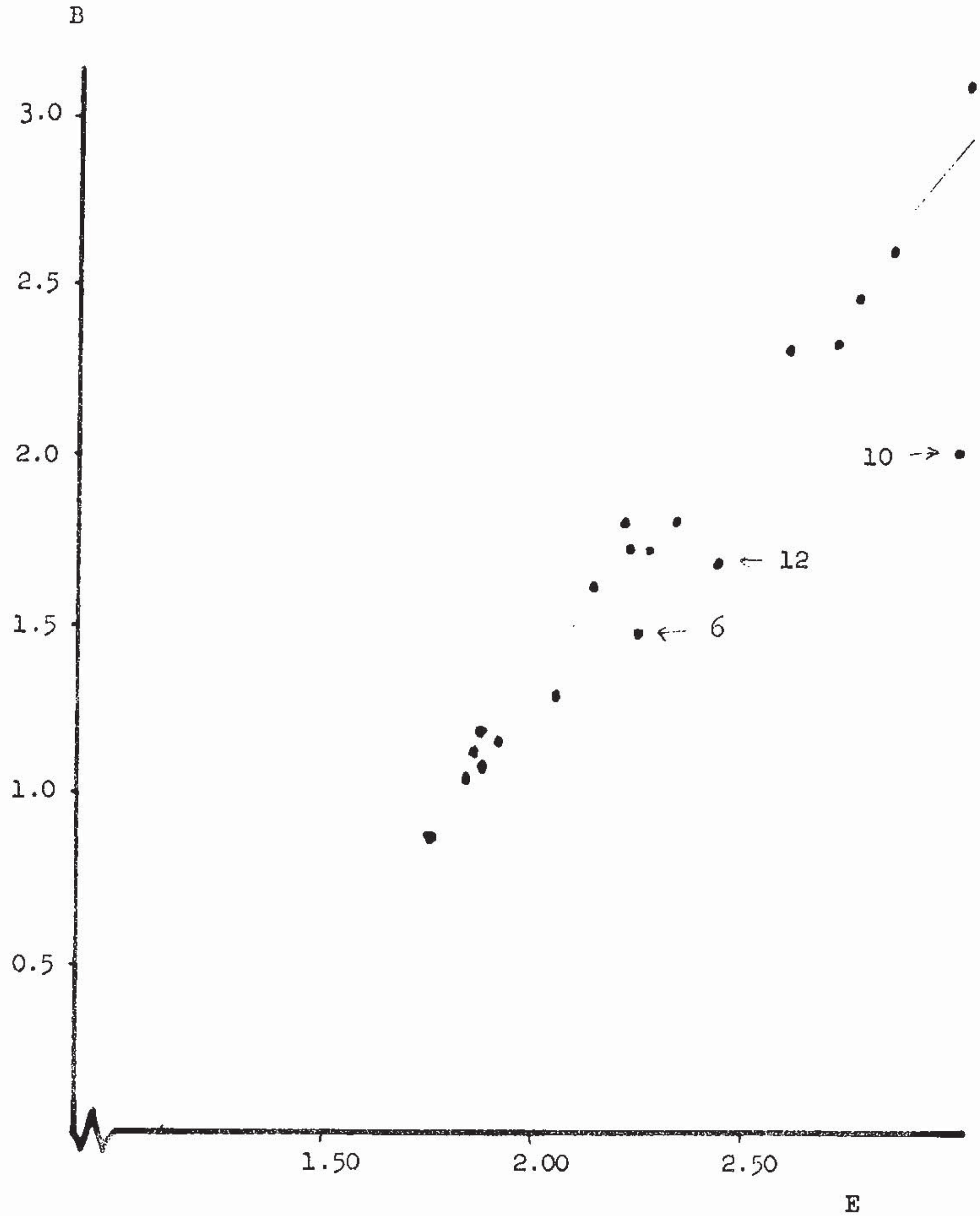

Fig. 26. - Predicted values of E and B for individual securities: subjective prediction experiment

Obviously these estimates overstated the counselor's true feelings concerning the distance of these points below the line. It is quite possible that the failure of these data to portray the absolute magnitudes of these distances correctly may have been the only major error which arose in the process of translating the investment counselor's beliefs into quantitative estimates. If this is the case, the consistency between the description of an arbitrager's views implied by our theory and the parameters obtained in the experiment provides a successful test of the positive theory.

C. Evidence on the Validity of the Theory

The theory of security market behavior presented here concerns the relationships among predicted values of various parameters. Since the test described in the previous section used ex ante values it constituted a direct test of the theory. Unfortunately, the majority of available data deals with ex post values, which describe not the predicted values of the parameters, but their actual magnitudes. Since prediction in the security market is an imperfect art, considerable differences between ex post and ex ante values of such parameters must occur; nonetheless, relationships which hold for ex ante values should also prove valid for ex post magnitudes, although with considerably less significance. This reduction in accuracy should be kept in mind in evaluating the tests reported in this section, all of which use ex post values as estimates of the corresponding ex ante magnitudes.

One implication of the theory concerns the relationship between average yield and standard deviation of yield for diversified portfolios: all such portfolios should lie along a straight line in the (σ, E) plane passing through the point at which $E = (1 + r)$ and $\sigma = 0$.

To test this implication, the performance of 23 open-end mutual funds was examined; this sample includes all common stock and balanced funds for which yield data were given in Wiesenberger's Investment Companies for all years from 1940 through 1959.[7]

Figure 27 shows the average yield and standard deviation of yield for the 23 mutual funds during the period 1940-51. The performance of two other large portfolios is also shown: the first, indicated by the point SP, comprises the common stocks which constitute the Standard and Poor's 90-stock index; the second, designated by point R, is a portfolio which includes equal amounts of the 96 industrial common stocks analyzed in this study. A line intersecting the horizontal axis at 1.03 (which corresponds to a 3% interest rate) has been drawn free-hand in Fig. 27; while there is considerable dispersion around this line, the data do not appear to be strongly inconsistent with the theory.

Figure 28 shows the performance of the 23 mutual funds during

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[7]With one exception: Affiliated Fund was excluded from the analysis since its operations during some of the years were unusual for a mutual fund. Yields for the 23 companies were measured in accordance with the rules specified in Investment Companies (New York: Arthur Wiesenberger and Company, 1946-1960).

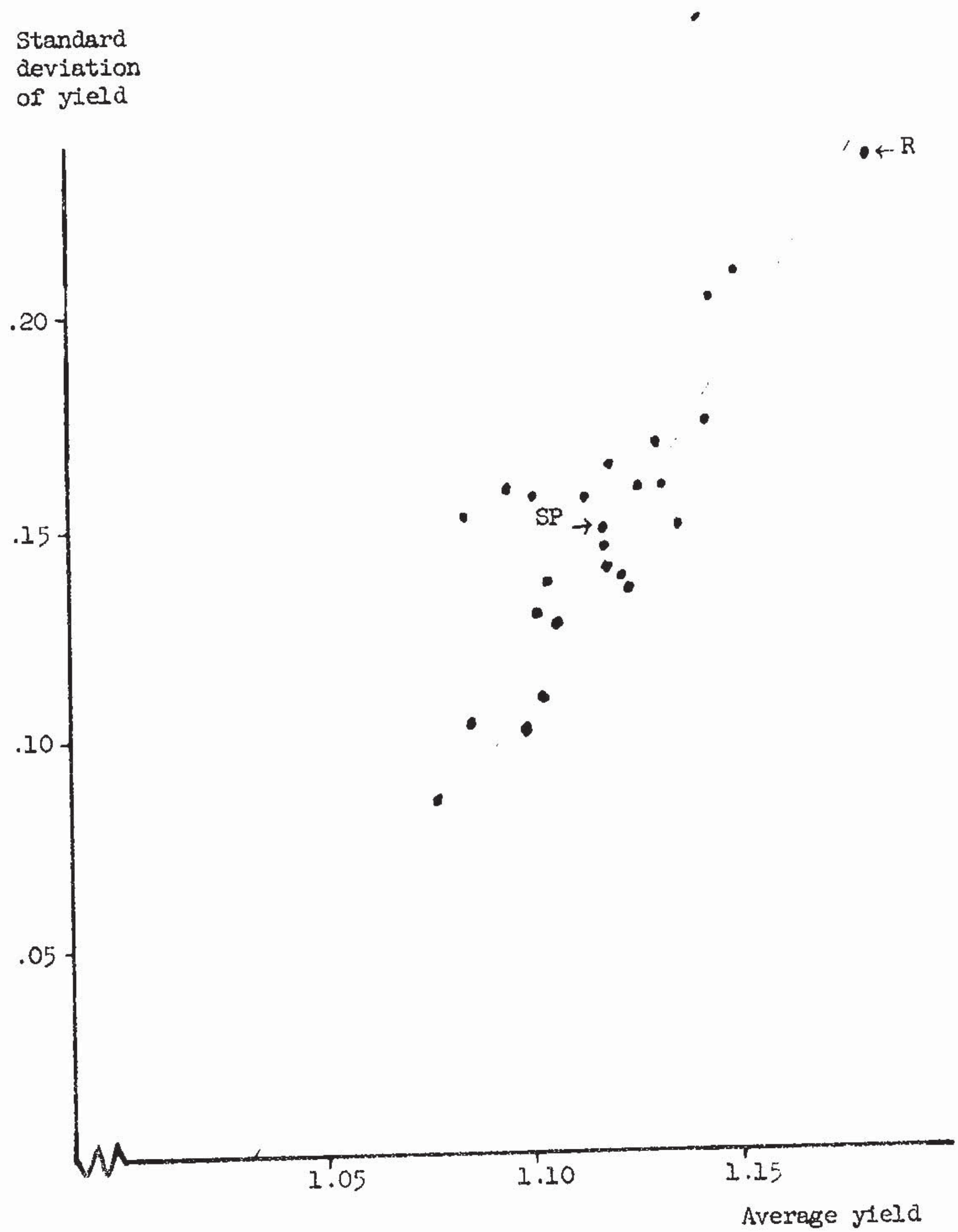

Fig. 27. - Average yield and standard deviation of yield: 23 investment companies and two indices, 1940-1951

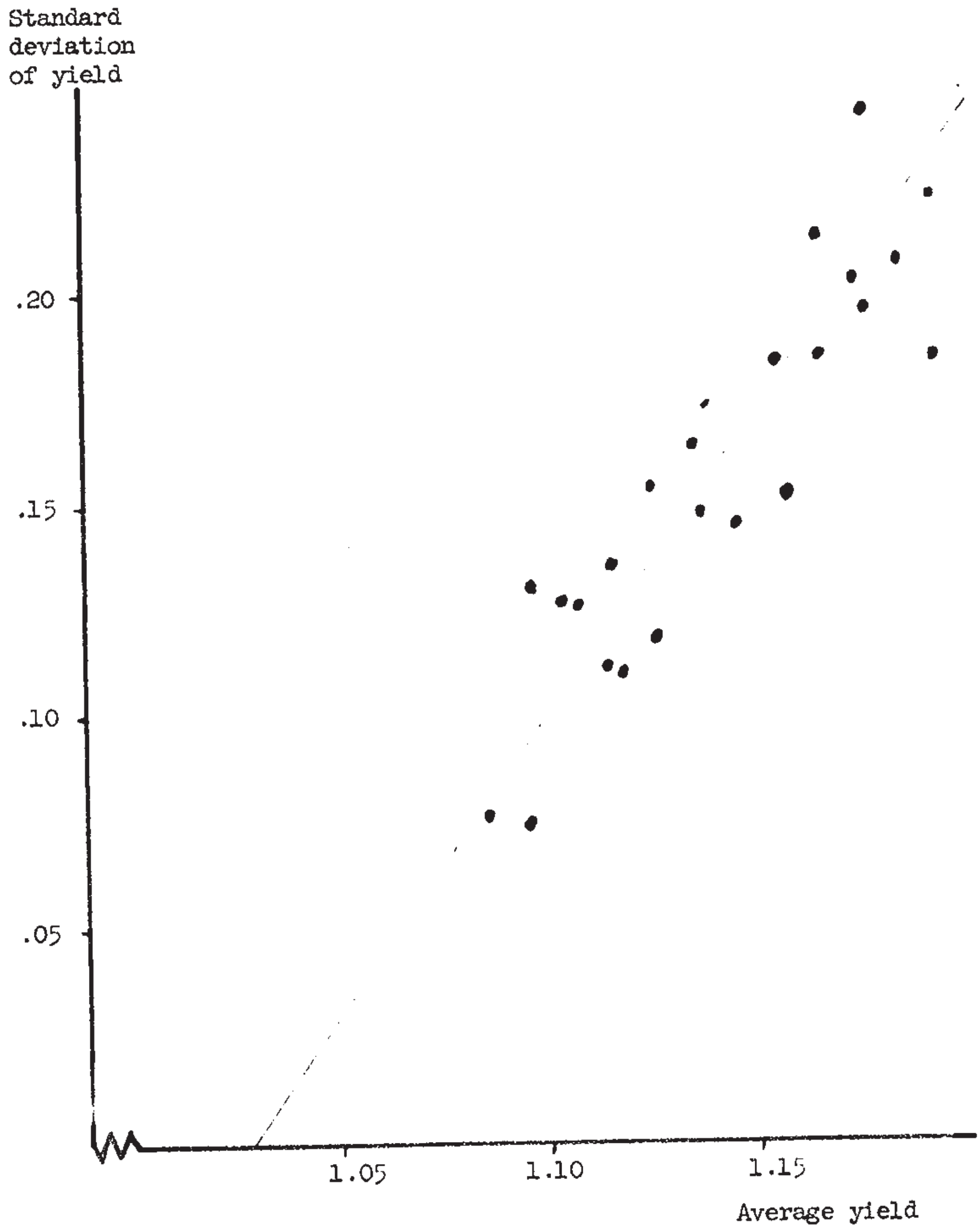

Fig. 28. - Average yield and standard deviation of yield: 23 investment companies, 1952-1959

the period 1952-59. Again a free-hand line has been drawn through 1.03 on the horizontal axis. These data also appear to be consistent with the implications of the theory.

A second implication of the theory concerns the relationship between B_i and E_i for individual securities; the model implies a linear relationship, with the value of E_i equal to (one plus the pure interest rate) when B_i is zero. To test this hypothesis, two periods were examined: 1940-51 and 1946-59. For each period, the correlation coefficient between B_i and E_i was calculated and two regression lines fitted by the least-squares technique.[8] Since both B_i and E_i are affected by price adjustments (the forces which bring about the implied relationship between these two parameters), the true relationship between B_i and E_i lies between the line obtained by the regression of B_i on E_i and that obtained by regressing E_i on B_i. Table 6 presents the results of the analysis.

The data in Table 6 are consistent with the hypothesis. The relationship between B_i and E_i is significant in both periods (a correlation coefficient of 0.26 would be significant at the 1% level). Moreover, the true line which relates B_i to E_i appears to

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[8]For the early period the parameters were estimated in the manner described earlier. For the second period a different measure was used for (I) in estimating the B_i parameters. The index adopted was Moody's 200-common stock index; yearly figures were based on annual averages of the weekly level of the index. Data are given in Moody's Industrial Manual, (New York: Moody's Investors Service, 1940-1960).

Table 6

RELATIONSHIP BETWEEN B_i AND E_i FOR
96 INDUSTRIAL COMMON STOCKS

	1940-51	1946-59
Correlation coefficient between B_i and E_i	0.511	0.283
Value of E_i when $B_i = 0$:		
(1) Based on regression of B_i on E_i	0.942	0.934
(2) Based on regression of E_i on B_i	1.122	1.133
(3) Arithmetic mean of (1) and (2)	1.032	1.034

cut the average-yield axis at nearly the pure interest rate.[9]

Another implication of the theory concerns the relationship between Q_i and E_i. In equilibrium, the average yield of a security should depend entirely on its B_i parameter; there is no necessary reason for correlation between the average yield of a security and its Q_i parameter. Nonetheless, since the security market is not always in equilibrium, the value of Q_i may influence the yield of a security; if a security becomes undervalued, the rapidity with which

[9] One method for fitting a straight line to data under these circumstances was proposed by Abraham Wald, "The Fitting of Straight Lines if Both Variables Are Subject to Error," The Annals of Mathematical Statistics, XI (September, 1940), 284-300. When applied to the data of the 1940-51 period, Wald's method gave a regression line which intersected the E-axis at a value of 1.056.

its price will increase depends on the extent to which investors will wish to add it to their portfolios. We have shown that the larger is Q_i, the smaller will be the amount of such a security included in efficient portfolios; thus securities with high Q_i parameters will move more slowly toward equilibrium, causing an observed positive correlation between Q_i and E_i. In addition, securities for which Q_i is high tend to have large values of B_i; thus Q_i and E_i are likely to be related due to the correlation between B_i and E_i. In spite of these factors, however, the correlation between B_i and E_i should prove more significant than that between Q_i and E_i; a contrary finding would clearly be inconsistent with the theory.

The evidence of the two periods is contradictory. In the first period, the correlation between Q_i and E_i exceeded that between B_i and E_i -- the simple correlation coefficients were 0.667 and 0.511. On the other hand, the data from the second period were consistent with the hypothesis; the simple correlation coefficient between B_i and E_i was 0.283, more than twice 0.130, the coefficient for the correlation between Q_i and E_i.

Few conclusions can be reached from the evidence presented in this section; although the majority of the data proved to be consistent with the theory, one important set clearly contradicted one of its major implications. However, these data indicate that it would be premature to reject the theory at this time; extensive empirical tests are required in order to assess its true value as a tool of positive economics.

VI. CONCLUSIONS

This dissertation has presented a simple model of the relationships among securities and examined its role in both positive and normative applications. This concluding chapter very briefly reviews the major conclusions to be drawn from the study. As indicated in the previous chapters, a number of the conclusions rest on the results of extremely limited testing; a few have not been subjected to any tests utilizing independent data. For these reasons the statements contained here must be considered extremely tentative and are best viewed as hypotheses which appear worthy of further tests. With these limitations in mind, we may recapitulate the major findings of the study.

One of the major advantages of the diagonal model is the simple form in which the associated portfolio-analysis problem can be stated. This simplified formulation greatly reduces the cost of applying the Markowitz technique; the machine program developed as part of this study is able to perform a complete portfolio analysis of 100 securities for a total cost of $5. Such a low figure greatly increases the probability that the value of portfolio analysis will exceed its cost.

Two normative applications of portfolio analysis have been examined. The first investigated the value of objective prediction techniques. It appears that when such techniques are utilized, portfolio analysis using estimates of the parameters of the diagonal model is not significantly inferior to an analysis which uses a complete set of estimates of the variances and covariances among securities; and since the former is much less expensive, it is likely to be the preferable technique. In general, objective prediction techniques are likely to prove valuable if a sufficiently large number of securities (several hundred) are analyzed. Efficient portfolios obtained in this manner appear to be as desirable as large portfolios of securities selected at random: those with similar average yields will, in general, have similar standard deviations of yield. Moreover, objective prediction techniques are likely to be superior with respect to one important characteristic. It appears to be possible, with the Markowitz technique, to predict with some accuracy the degree of conservatism of the efficient portfolios: rankings based on predicted standard deviation of yield agree quite well with those based on actual standard deviation of yield.

The second normative application of the technique concerned the feasibility of using the Markowitz technique with subjective predictions of security performance. The experiment devised for this test indicated that a substantial amount of quantitative information concerning the predictions of an investment counselor could be obtained for use in a subsequent portfolio analysis. While it is likely that the estimates obtained for this purpose will contain some errors concerning the magnitudes of differences among securities, these errors will be of little consequence if a sufficiently large number of securities are analyzed. However, it is unlikely that portfolios chosen with the portfolio-analysis technique will prove to be markedly superior to those chosen by an experienced investment analyst without benefit of the technique. The practical value of the analysis probably lies in the division of labor it facilitates between the security analyst and less specialized investment advisors.

Although normative applications of the Markowitz approach to portfolio analysis remain of interest, its greatest contribution may be in the field of positive economics. The theory of security market behavior developed in this study appears to have considerable predictive value. While more sophisticated models will undoubtedly exhibit greater empirical validity, it seems likely that the best theory of security market behavior will be developed, as was this one, from the assumption that investors seek to maximize some utility function. The Markowitz formulation represents the process of investment selection in just such terms; for this reason it is likely to be a major element in future successful theories of security market behavior.

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APPENDIX A

PORTFOLIO ANALYSIS WITH THE DIAGONAL MODEL: THE SOLUTION TECHNIQUE

This appendix presents a solution technique for the portfolio analysis problem associated with the diagonal model. Markowitz's critical line method[1] is used for the computations with the special characteristics of the diagonal model taken into account in order to simplify the procedure. The presentation follows Markowitz in order to facilitate comparison. All variables not defined in this Appendix have the meanings assigned in Chapter II.

1. Basic Inputs

In Markowitz's presentation, (n) represents the total number of securities, both real and artificial. For convenience we will use (n) to represent the number of actual securities, letting the $(n+1)^{st}$ security be the market index; thus all matrices and vectors will be of size (n+1).

We begin with the matrix of covariances, C. This will be an (n+1) by (n+1) matrix with zeroes everywhere except along the main diagonal, where the Q_i's will be found. Thus, if c_{ij} represents the element in the i^{th} row and j^{th} column of C:

$$c_{ij} = Q_i \quad \text{if } i = j$$

$$c_{ij} = 0 \quad \text{if } i \neq j$$

The next input is U, the column vector of expected returns. This vector has (n+1) elements; u_i, the i^{th} element, equals A_i, the first parameter in the basic equation of the diagonal model.

[1] Markowitz, Portfolio Selection. . . , pp. 172-186.

Another set of inputs specifies the constraint equations. A is a 2 by (n+1) matrix and b a two-element vector. The constraints are:

$$Ax = b.$$

The first row of the A-matrix is used to require the sum of the amounts invested in the first n securities to equal one. Let a_{1j} be the element in the j^{th} column of the first row of A, and b_1 the first element in b. Then:

$$a_{1j} = 1 \qquad \text{if } j \leqq n$$
$$a_{1,n+1} = 0$$
$$b_1 = 1.$$

The second row of A is used to define X_{n+1}. The elements take the following values:

$$a_{2j} = B_j \qquad \text{if } j \leqq n$$
$$a_{2,n+1} = -1$$
$$b_2 = 0$$

2. Matrices Used in The Computation

The critical line method requires that the basic inputs described in the previous section be combined into new matrices. This section describes the formation of these matrices and specifies their elements.

The basic matrix in the computation is M, defined as follows:

$$M = \begin{bmatrix} C & A' \\ A & 0 \end{bmatrix}$$

M is an (n+3) by (n+3) matrix. Let m_{ij} be the element in the i^{th} row and j^{th} column of M. Then from the conditions specified for C and A:

$$m_{ij} = Q_i \text{ for } i = j \leqq n+1$$

$$m_{n+2,j} = 1 \text{ for } j \leqq n$$

$$m_{n+3,j} = B_j \text{ for } j \leqq n$$

$$m_{n+3,n+1} = -1$$

$$m_{i,n+2} = 1 \text{ for } i \leqq n$$

$$m_{i,n+3} = B_i \text{ for } i \leqq n$$

$$m_{n+1,n+3} = -1$$

All other elements are zero.

R is an (n+3) element column vector. Its first n+1 elements are zeros, the last two are those of the vector b. If r_i is the element in the i^{th} row of R:

$$r_i = 0 \text{ for } i \leqq n+1$$

$$r_{n+2} = 1$$

$$r_{n+3} = 0$$

S is another (n+3) element column vector. Its first n+1 elements are those of the vector U; the last two elements are zero. If s_i is the element in the i^{th} row of S:

$$s_i = A_i \text{ for } i \leqq n+1$$

$$s_i = 0 \text{ for } i > n+1$$

These three matrices are summarized in the following diagrams.

$$
M = \begin{array}{c|ccccc|c|cc}
\text{Row} \backslash \text{Column} & 1 & 2 & 3 & \cdots & n & n+1 & n+2 & n+3 \\
\hline
1 & Q_1 & & & & & & 1 & B_1 \\
2 & & Q_2 & & & & & 1 & B_2 \\
3 & & & Q_3 & & & & 1 & B_3 \\
\vdots & & & & \ddots & & & \vdots & \vdots \\
n & & & & & Q_n & & 1 & B_n \\
n+1 & & & & & & Q_{n+1} & 0 & -1 \\
\hline
n+2 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 \\
n+3 & B_1 & B_2 & B_3 & \cdots & B_n & -1 & 0 & 0
\end{array}
$$

$$
R = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}
\qquad
S = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \\ A_{n+1} \\ 0 \\ 0 \end{bmatrix}
$$

3. Inverting $\tilde{M}$

In each step of the critical line method a new matrix, $\tilde{M}$, is formed by replacing some of the positive values of M with zeroes or ones; this is followed by the inversion of $\tilde{M}$. In this section we develop equations for inverting $\tilde{M}$ for the general case in which all elements which can take on non-zero values in M are also non-zero in $\tilde{M}$; the results can then be applied to any $\tilde{M}$ formed in the analysis.

Let v_{ij} be the element in the i^{th} row and j^{th} column of the inverse of $\tilde{M}$. Then the values for column j of this matrix can be found by solving the n+3 simultaneous equations:

$$M \begin{bmatrix} v_{1j} \\ v_{2j} \\ v_{3j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_{nj} \\ v_{n+1,j} \\ v_{n+2,j} \\ v_{n+3,j} \end{bmatrix} = \begin{bmatrix} z_{1j} \\ z_{2j} \\ z_{3j} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_{nj} \\ z_{n+1,j} \\ z_{n+2,j} \\ z_{n+3,j} \end{bmatrix}$$

Where: $z_{ij} = 1$ if $i = j$

and $z_{ij} = 0$ if $i \neq j$

In order to account for the possibility that values may be made zero when M is converted to $\tilde{M}$, we need to replace the values of the $(n+2)^{nd}$ row and column with more general terms. We shall let D_k represent the element in the k^{th} column of the $(n+2)^{nd}$ row (and, therefore, also, the element in the $(n+2)^{nd}$ column of the k^{th} row). In addition, we will use the symbol B_{n+1} for $m_{n+3,n+1}$ ($= m_{n+1,n+3}$) to account for the possibility that these values might become zero.

With these changes, the equations for the elements of the j^{th} column of the inverse of $\widetilde{M}$ becomes:

$$
\begin{array}{c|cccccc|}
\text{Column} \backslash \text{Row} & 1 & \cdots & n & n+1 & n+2 & n+3 \\
\hline
1 & Q_1 & & & & D_1 & B_1 \\
 & & \ddots & & & \vdots & \vdots \\
n & & & Q_n & & \vdots & \vdots \\
n+1 & & & & Q_{n+1} & D_{n+1} & B_{n+1} \\
n+2 & D_1 & D_2 & \cdots & D_{n+1} & 0 & 0 \\
n+3 & B_1 & B_2 & \cdots & B_{n+1} & 0 & 0
\end{array}
\begin{bmatrix} v_{1j} \\ \vdots \\ v_{nj} \\ v_{n+1,j} \\ v_{n+2,j} \\ v_{n+3,j} \end{bmatrix}
=
\begin{bmatrix} z_{1j} \\ \vdots \\ z_{nj} \\ z_{n+1,j} \\ z_{n+2,j} \\ z_{n+3,j} \end{bmatrix}
$$

The first n+1 equations are:

$$Q_1 v_{1j} + D_1 v_{n+2,j} + B_1 v_{n+3,j} = z_{1j}$$

$$Q_2 v_{2j} + D_2 v_{n+2,j} + B_2 v_{n+3,j} = z_{2j}$$

$$\vdots$$

$$Q_{n+1} v_{n+1,j} + D_{n+1} v_{n+2,j} + B_{n+1} v_{n+3,j} = z_{n+1,j}$$

Rewriting:

$$v_{1j} = z_{1j}/Q_1 - (D_1/Q_1) v_{n+2,j} - (B_1/Q_1) v_{n+3,j}$$

$$v_{2j} = z_{2j}/Q_2 - (D_2/Q_2) v_{n+2,j} - (B_2/Q_2) v_{n+3,j}$$

$$\vdots \qquad \vdots$$

$$v_{n+1,j} = z_{n+1,j}/Q_{n+1} - (D_{n+1}/Q_{n+1})\, v_{n+2,j} - (B_{n+1}/Q_{n+1}) v_{n+3,j}$$

Equation n+2 is:

$$D_1 v_{1j} + D_2 v_{2j} + \ldots + D_{n+1} v_{n+1,j} = z_{n+2,j}$$

Each of the v_{ij} elements in this equation can be replaced with an expression in $v_{n+2,j}$ and $v_{n+3,j}$ by substituting the first (n+1) equations. The resulting equation is:

$$z_{n+2,j} = \sum_{i=1}^{n+1} \left(\frac{D_i z_{ij}}{Q_i} \right) - \left[\sum_{i=1}^{n+1} \left(\frac{D_i^2}{Q_i} \right) \right] v_{n+2,j} - \left[\sum_{i=1}^{n+1} \left(\frac{D_i B_i}{Q_i} \right) \right] v_{n+3,j}$$

Since z_{ij} equals zero if $i \neq j$ and 1 otherwise, the first term is simply equal to (D_j/Q_j). If j is between 1 and n+1; when j = (n+2) or (n+3), this term becomes zero. Letting:

(A1) $$L = \sum_{i=1}^{n+1} \frac{D_i^2}{Q_i}$$

(A2) $$N = \sum_{i=1}^{n+1} \frac{D_i B_i}{Q_i}$$

The (n+2) equation can be written:

$$z_{n+2,j} = \frac{D_j}{Q_j} - L v_{n+2,j} - N v_{n+3,j}$$

Note that the values of L and N are the same for every column of the inverse. Rewriting equation (n+2) to express $v_{n+3,j}$ as a function of $v_{n+2,j}$, we have:

(A3) $$v_{n+3,j} = \frac{1}{N} \left[\frac{D_j}{Q_j} - z_{n+2,j} \right] - \frac{L}{N} \left(v_{n+2,j} \right)$$

The $(n+3)^{rd}$ equation is:

$$B_1 v_{1j} + B_2 v_{2j} + \ldots + B_{n+1} v_{n+1,j} = z_{n+3,j}$$

Again we substitute expressions in $v_{n+2,j}$ and $v_{n+3,j}$ for the v_{ij} elements.

$$z_{n+3,j} = \sum_{i=1}^{n+1} \left(\frac{B_i z_{ij}}{Q_i} \right) - \left[\sum_{i=1}^{n+1} \left(\frac{B_i D_i}{Q_i} \right) \right] v_{n+2,j} - \left[\sum_{i=1}^{n+1} \left(\frac{B_i^2}{Q_i} \right) \right] v_{n+3,j}$$

Since z_{ij} equals 1 when $i = j$ and zero when $i \neq j$, the first term is merely B_j/Q_j when j is between 1 and n+1; if $j = n+2$ or $n+3$, this term becomes zero. The second term has already been defined as N (Eq. A2). Letting

(A4) $$P = \sum_{i=1}^{n+1} \frac{B_i^2}{Q_i}$$

This equation can be written:

$$z_{n+3,j} = \frac{B_j}{Q_j} - Nv_{n+2,j} - Pv_{n+3,j}$$

Substituting Eq. (A3) for the value of $v_{n+3,j}$, we have:

(A5) $$v_{n+2,j} = \left[\frac{N}{PL - N^2} \right] \left[z_{n+3,j} + \frac{P}{N} \left(\frac{D_j}{Q_j} - z_{n+2,j} \right) - \frac{B_j}{Q_j} \right]$$

This completes the solution of the set of simultaneous equations. The steps required to find the value of any element in column j of the inverse of $\acute{M}$ are:

a. Compute the values of L, N, and P according to equations (A1), (A2), and (A4). Note that these values need to be computed only once for a particular $\acute{M}$.

b. Compute the value of $v_{n+2,j}$ according to formula (A5).

c. Compute the value of $v_{n+3,j}$ by substituting the value calculated in (b) into Eq. (A3).

d. Compute the values of desired elements in the column. The element in the i^{th} row of column j has a value given by the formula:

(A6) $$v_{ij} = z_{ij}/Q_i - (D_i/Q_i)v_{n+2,j} - (B_i/Q_i)v_{n+3,j}$$

4. The Equation of a Critical Line

A key element in Markowitz's solution technique is the concept of the critical line. Any portfolio can be characterized in terms of the securities which it contains (these securities are "in") and those which it does not contain (the latter are "out".) The preliminary steps for computing a critical line are:

a. For each security which is "out", replace all elements of the appropriate row and column of M with zero, except the element at the intersection, which becomes one. (This process is called replacement with a "unit cross".) The resulting matrix is $\breve{M}$.

b. Replace all elements of (S) corresponding to securities which are "out" with zero. The resulting vector is $(\breve{S})$.

The equation of the critical line is then:

$$\begin{bmatrix} X \\ \lambda \end{bmatrix} = (\breve{M})^{-1}R + (\breve{M})^{-1}\breve{S}\lambda_E$$

where X represents an (n+1)-element vector and λ the two-element vector $[\lambda_1 \lambda_2]$.

The equation for the critical line is quite simple with the diagonal model. The term $(\breve{M})^{-1}R$ becomes:

$$\begin{bmatrix} \\ \\ & (\breve{M})^{-1} & \\ \\ \\ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The product is the $(n+2)^{nd}$ column of the inverse of $\tilde{M}$. Let T be an (n+3)-element column vector equal to $(M)^{-1}R$. Then the equation of the critical line becomes:

$$\begin{bmatrix} X \\ \lambda \end{bmatrix} = T + (\tilde{M})^{-1}\tilde{S}\lambda_E$$

and, if t_i is the i^{th} element of T:

$$t_i = v_{i,n+2}$$

The elements of T can be shown to be:

$$t_i = K_1\left(\frac{D_i}{Q_i}\right) + K_2\left(\frac{B_i}{Q_i}\right) \quad \text{for i from 1 to n+1}$$

$$t_{n+2} = -K_1$$

$$t_{n+3} = -K_2$$

where:

$$K_1 = \left[\frac{N}{PL-(N)^2}\right] \left[\frac{P}{N}\right]$$

and

$$K_2 = \frac{1 - K_1 L}{N}$$

The remaining vector of the equation can also be summarized. Let:

$$U = (\tilde{M})^{-1}\tilde{S}$$

It can be shown that the elements of u_i are:

$$u_i = \frac{A_i}{Q_i} - S_{AD}\left[K_1\left(\frac{D_i}{Q_i}\right) + K_2\left(\frac{B_i}{Q_i}\right)\right] - S_{AB}\left[K_3\left(\frac{D_i}{Q_i}\right) + K_4\left(\frac{B_i}{Q_i}\right)\right] \quad \text{for i from 1 to n+1}$$

$$u_{n+2} = K_1 S_{AD} + K_3 S_{AB}$$

$$u_{n+3} = K_2 S_{AD} + K_3 S_{AB}$$

where:

$$K_3 = - \frac{N}{PL - N^2}$$

$$K_4 = - \frac{K_3 L}{N}$$

$$S_{AD} = \sum_{i=1}^{n+1} \frac{A_i D_i}{Q_i} \quad \text{(for all i which are "in")}$$

$$S_{AB} = \sum_{i=1}^{n+1} \frac{A_i B_i}{Q_i} \quad \text{(for all i which are "in")}$$

5. Obtaining Corner Portfolios

The major steps in the critical line method for portfolio analysis follow.

a. Select the security with the highest expected yield (ties are broken arbitrarily). This is the first corner portfolio.

b. Compute the critical line of the portfolio for which this security is "in".

c. Compute the value of λ_E at the intersection of this critical line and every critical line associated with a portfolio for which this security and one other are "in". The portfolio which gives the highest value of λ_E is the second corner portfolio.

d. Compute the value of λ_E at the intersection of the critical line associated with the new portfolio and:
   1. each critical line associated with a portfolio which includes all but one of the securities which are in this portfolio, and
   2. each critical line associated with a portfolio which includes all securities which are in this portfolio plus one additional security.

e. The portfolio which gives the highest value of λ_E (less than that found at the previous intersection) is the next corner portfolio.

f. If $\lambda_E \leq 0$, the solution is complete. If not, repeat steps (d) through (e) using the new corner portfolio.

This section derives the formulae from which the intersections of critical lines can be computed.

Intersections of the type described in step (d1) are easily computed. From the equation of a critical line:

$$X_i = t_i + u_i\lambda_E$$

The value of λ_E at which the i^{th} security drops out of the portfolio can be found by setting $X_i = 0$:

(A6)
$$\lambda_E = -\frac{t_i}{u_i}$$

Intersections of the type described in step (d2) are somewhat more difficult. Let the formula for the critical line associated with a portfolio for which security K is out be represented by:

$$\breve{M} \begin{bmatrix} X \\ \lambda \end{bmatrix} = R + \breve{S}\lambda_E$$

and the formula for the critical line associated with a portfolio containing the same securities plus security K be represented by:

$$\breve{M}_K \begin{bmatrix} X \\ \lambda \end{bmatrix} = R + \breve{S}_K\lambda_E.$$

Since the vector R is the same in both cases, we have:

$$R = \breve{M} \begin{bmatrix} X \\ \lambda \end{bmatrix} - \breve{S}\lambda_E = \breve{M}_K \begin{bmatrix} X \\ \lambda \end{bmatrix} - \breve{S}_K\lambda_E$$

Rewriting:

$$\left[\widetilde{M}_K - \widetilde{M}\right]\begin{bmatrix} X \\ \lambda \end{bmatrix} = \left[\widetilde{S}_K - \widetilde{S}\right]\lambda_E$$

But $\widetilde{S}_K$ is merely the vector $\widetilde{S}$ except that instead of a zero in the k^{th} row, $\widetilde{S}_K$ has the value A_k. Thus the vector $\left[\widetilde{S}_K - \widetilde{S}\right]$ has zeroes in every position but the k^{th} row, where the value of A_k appears.

$\widetilde{M}_K$ is merely $\widetilde{M}$ with the unit cross of row and column k replaced with the values originally in M. We can disregard all rows but the k^{th} since they give us no information: each will have an element in the k^{th} column only, all others being zero. It can be seen that each row but the k^{th} thus reduces to the equation: $0 \cdot X_k = 0$.

Consider the k^{th} row of the matrix $\left[\widetilde{M}_K - M\right]$. In $\widetilde{M}$ all elements of the row are zero but the element in the k^{th} column, which is 1. In $\widetilde{M}_K$ all elements of the row have the original values from M. Thus, except for the element in the k^{th} column, all elements have the values of the k^{th} row of M; the element in the k^{th} column is merely the original value less 1. The equation for the k^{th} row becomes:

$$\left[m_{k1}, m_{k2}, \ldots, (m_{k,k} - 1), \ldots, m_{k,n+3}\right]\begin{bmatrix} X \\ \lambda \end{bmatrix} = A_k\lambda_E$$

But the intersection of the two critical lines must be on the initial line, so we can insert the solution for the initial line in place of the vector:

$$\left[m_{k1}, m_{k2}, \ldots, (m_{k,k} - 1), \ldots, m_{k,n+3}\right]\left[T + U\lambda_E\right] = A_k\lambda_E$$

Solving for the value of λ_E:

$$\lambda_E = \frac{\left[m_{k1}, m_{k2}, \ldots, (m_{kk} - 1), \ldots, m_{k,n+3}\right] T}{A_k - \left[m_{k1}, m_{k2}, \ldots, (m_{kk} - 1), , m_{k,n+3}\right] U}$$

From the properties of the diagonal model we know that m_{k1} through $m_{k,n+1}$ are all zero except m_{kk} and thus can be disregarded. Further, the k^{th} element of T and the k^{th} element of U are also zero since X_k is out on the critical line for which T and U are calculated. Thus $(m_{kk} - 1)$ can also be disregarded. The formula for λ_E thus becomes:

$$\lambda_E = \frac{m_{k,n+2}\, t_{n+2} + m_{k,n+3}\, t_{n+3}}{A_k - m_{k,n+2}\, u_{n+2} - m_{k,n+3}\, u_{n+3}}$$

We can simplify the formula a little further. Since only the addition of real securities will be considered, k will range from 1 to n. Thus $m_{k,n+2} = 1$ for all k to be considered. Similarly, $m_{k,n+3} = B_k$. Thus the formula simplifies to:

(A7) $$\lambda_E = \frac{t_{n+2} + B_k\, t_{n+3}}{A_k - u_{n+2} - B_k\, u_{n+3}}$$

Formulae (A6) and (A7) complete the solution technique. Appendix B describes a computer program which uses this method.

APPENDIX B

THE DIAGONAL MODEL PORTFOLIO ANALYSIS CODE

This appendix describes a machine program, written in the FORTRAN language, for performing portfolio analysis based on the diagonal model, using the solution technique described in Appendix A.

The inputs required by the program are:

1. Title card: any valid Hollerith characters, (blanks included) in columns 2 through 72.

2. One card indicating the number of securities (n). Punching should be in columns 1 through 5, with the number right-justified and no decimal point included.

3. One card containing information about the index used for the problem.

   Columns 7-35: name of index: any valid Hollerith characters, blanks included.
   Columns 40-49: expected value of the index (A_{n+1}). Decimal point must be included.
   Columns 60-69: variance of the index (Q_{n+1}). Decimal point must be included.

   Leave all other columns blank.

4. (N) cards, one for each security to be considered. Each card must contain the following information:

   Columns 7-35: name of security: any valid Hollerith characters, blanks included.
   Columns 40-49: (A_i) parameter for the security. Decimal point must be punched.
   Columns 50-59: (B_i) parameter for the security. Decimal point must be punched.
   Columns 60-69: (Q_i) parameter for the security. Decimal point must be punched.

   Leave all other columns blank.

The outputs from the program are:

1. Data listing

    a. Title of problem
    b. Input data
        1. Lists all inputs
        2. Assigns numbers (in order) to securities for identification of the remainder of output.

2. Corner portfolio characteristics: each corner portfolio appears on a separate page, with the highest-yield portfolio first. Each such page contains the following information:

    a. Corner portfolio number
    b. Portfolio characteristics
        1. E: expected yield
        2. STD DEV: standard deviation of yield
        3. dV/dE: slope of E-V curve at this point
    c. Portfolio composition -- The amounts of each of the N securities entering the portfolio are shown in tabular form. The left-hand column gives the first digit of the security number; the top row gives the second digit.

The only restrictions on the program concern the number of inputs. No more than 1,999 securities can be analyzed in one run. Running time will depend on the number of corner portfolios: an expected value is 30 seconds per 100 securities. For safety, a planning factor of one minute per 100 securities is recommended. A listing of the FORTRAN source program follows.

```
      DIMENSION A(2000),B(2000),Q(2000),ABAR(2000),DBAR(2000),
     1 BBAR(2000),ZINOUT(2000),X(2000),T(2000),U(2000)
100   FORMAT (I5)
101   FORMAT (1HOI3,34H                                 3F10.5)
102   FORMAT (44HO NO  SECURITY                              A
     1 25H          B          Q                )
103   FORMAT (1HOI10,5X,10F10.5 )
104   FORMAT ( 40H1                   FIRST CORNER PORTFOLIO      )
105   FORMAT (23HO                    E=F10.5,
     1 23H              STD DEV=F10.6,
     2 22H               DV/DE=F10.6)
106   FORMAT (72H1
     1                                            )
107   FORMAT (28H1  CORNER PORTFOLIO NUMBER  I3 )
108   FORMAT ( 40H1                  LAST CORNER PORTFOLIO       )
109   FORMAT ( 44HOSECURITY NUMBER    1        2        3
     1 62H     4        5        6        7        8
     2 30H9        10                   )
      READ INPUT TAPE 5, 106
      WRITE OUTPUT TAPE 6, 106
      READ INPUT TAPE 5, 100, NSEC
      NP1 = NSEC + 1
      NP2 = NSEC + 2
      NP3 = NSEC + 3
      NUMB = 1
      WRITE OUTPUT TAPE 6, 102
      READ INPUT TAPE 5, 101, M, A(NP1), B(NP1), Q(NP1)
      WRITE OUTPUT TAPE 6, 101, M, A(NP1), B(NP1), Q(NP1)
      DO 200 I = 1, NSEC
      READ INPUT TAPE 5, 101, M, A(I), B(I), Q(I)
200   WRITE OUTPUT TAPE 6, 101, I, A(I), B(I), Q(I)
      B(NP1) = - 1.0
      DO 201  I = 1, NP1
      ABAR (I) = A(I) / Q(I)
      BBAR(I) = B(I) / Q(I)
201   DBAR(I) = 1.0 / Q(I)
      DBAR (NP1) = 0.0
      ZMAXE = 0.0
      DO 210 I = 1, NSEC
      E = A(I) + B(I) * A(NP1)
      IF ( E - ZMAXE ) 210, 210, 203
203   ZMAXE = E
      KMAXE = I
210   CONTINUE
      DO 300 I = 1, NSEC
300   ZINOUT (I) = 0.0
      ZINOUT (KMAXE ) = 1.0
      ZINOUT (NP1) = 1.0
      ZL = DBAR (KMAXE)
      ZN = DBAR(KMAXE) * B(KMAXE)
```

```
      ZP = B(KMAXE)**2 / Q(KMAXE) + 1.0 / Q(NP1)
      SUMAD = A(KMAXE) * DBAR(KMAXE)
      SUMAB = A(KMAXE) * BBAR(KMAXE) + A(NP1) *BBAR(NP1)
      DO 350 I = 1, NP3
      T(I) = 0.0
  350 U(I) = 0.0
      T(KMAXE) = 1.0
      T (NP1) = B ( KMAXE )
      ZK1 = ZP / ( ZP*ZL - ZN**2 )
      ZK2 = - ZN / ( ZP*ZL - ZN**2 )
      ZK3 = 1.0 / ZN
      ZK4 = - (ZK1*ZL ) / ZN
      ZK5 = - (ZK2 * ZL ) / ZN
      ZK6 = ZK3 + ZK4
      T(NP3) = - ZK6
      T(NP2) = - ZK1
      U(NP3) = ZK6 * SUMAD + ZK5 * SUMAB
      U(NP2) = ZK1 * SUMAD + ZK2 * SUMAB
      ZLMAX = 0.0
      KMAXP = KMAXE + 1
      KMAXM = KMAXE - 1
      DO 420 I = 1, KMAXM
      ZLAM = (T(NP2)+B(I)*T(NP3))/(A(I)-U(NP2)-B(I)*U(NP3))
      IF (ZLAM - ZLMAX) 420, 420, 410
  410 ZLMAX = ZLAM
      KCHG = I
  420 CONTINUE
      DO 440 I = KMAXP, NSEC
      ZLAM = (T(NP2)+B(I)*T(NP3))/(A(I)-U(NP2)-B(I)*U(NP3))
      IF ( ZLAM - ZLMAX) 440, 440, 430
  430 ZLMAX = ZLAM
      KCHG = I
  440 CONTINUE
      ZLAME = ZLMAX
      WRITE OUTPUT TAPE 6, 104
      E = 0.0
      VAR = 0.0
      DO 460 I = 1, NP1
      X(I) = T(I) + U(I) * ZLAME
      VAR = VAR + Q(I) * X(I) ** 2
  460 E = E + X(I) * A(I)
      STDEV = VAR ** .5
      Z2LAM = 2.0 * ZLAME
      WRITE OUTPUT TAPE 6, 105,E,STDEV, Z2LAM
      WRITE OUTPUT TAPE 6, 109
      IND = 0
  461 IP1 = IND + 1
      IF (NSEC - IND - 10) 462, 462, 463
  462 WRITE OUTPUT TAPE 6, 103, IND, (X(I), I = IP1, NSEC )
      GO TO 500
```

```
463  IP10 = IND + 10
     WRITE OUTPUT TAPE 6, 103, IND, (X(I), I = IP1, IP10 )
     IND = IND + 10
     GO TO 461
499  IF (ZINOUT (KCHG)) 500, 500, 600
500  ZL = ZL + DBAR (KCHG)
     ZN = ZN + BBAR (KCHG)
     ZP = ZP + B(KCHG)**2 / Q(KCHG)
     SUMAD = SUMAD + A(KCHG) * DBAR(KCHG)
     SUMAB = SUMAB + A(KCHG) * BBAR(KCHG)
     ZINOUT (KCHG) = 1.0
     GO TO 700
600  ZL = ZL - DBAR (KCHG)
     ZN = ZN - BBAR (KCHG)
     ZP = ZP - B(KCHG)**2 / Q(KCHG)
     SUMAD = SUMAD - A(KCHG) * DBAR(KCHG)
     SUMAB = SUMAB - A(KCHG) * BBAR (KCHG)
     ZINOUT (KCHG) = 0.0
700  ZK1 = ZP / (ZP*ZL - ZN**2 )
     ZK2 = - ZN / ( ZP*ZL - ZN**2 )
     ZK3 = 1.0 / ZN
     ZK4 = - (ZK1*ZL ) / ZN
     ZK5 = - (ZK2 * ZL ) / ZN
     ZK6 = ZK3 + ZK4
     T(NP3) = - ZK6
     T(NP2) = - ZK1
     U(NP3) = ZK6 * SUMAD + ZK5 * SUMAB
     U(NP2) = ZK1 * SUMAD + ZK2 * SUMAB
     DO 810 I = 1, NP1
     IF (ZINOUT(I)) 801, 801, 802
801  T(I) = 0.0
     U(I) = 0.0
     GO TO 810
802  T(I) = ZK1* DBAR(I) + ZK6 * BBAR(I)
     U(I) = ABAR(I)-SUMAD* (ZK1*DBAR(I)+ZK6*BBAR(I))
    1 - SUMAB * (ZK2* DBAR(I) + ZK5 * BBAR(I))
810  CONTINUE
     ZLMAX = 0.0
     KCHGX = KCHG
     DO 950 I = 1, NSEC
     IF (ZINOUT(I)) 910, 910, 900
900  ZLAM = - T(I) / U(I)
     IF (ZLAM - ZLAME ) 901, 950, 950
901  IF (ZLAM - ZLMAX) 950, 950, 903
903  IF ( I - KCHGX ) 902, 950, 902
902  ZLMAX = ZLAM
     KCHG = I
     GO TO 950
910  ZLAM = (T(NP2)+B(I)*T(NP3)) / (A(I) - U(NP2) - B(I)*U(NP3))
     IF (ZLAM - ZLAME ) 911, 950, 950
```

```
911  IF (ZLAM - ZLMAX) 950, 950, 913
913  IF (I - KCHGX) 912, 950, 912
912  ZLMAX = ZLAM
     KCHG = I
950  CONTINUE
     IF (ZLMAX ) 960, 960, 970
960  E = 0.0
     VAR = 0.0
     DO 961 I = 1, NP1
     VAR = VAR + Q(I) * T(I) **2
961  E = E + T(I) * A(I)
     STDEV = VAR ** .5
     WRITE OUTPUT TAPE 6, 103
     ZPDQ = 0.0
     WRITE OUTPUT TAPE 6, 105, E, STDEV, ZPDQ
     WRITE OUTPUT TAPE 6, 109
     IND = 0
962  IP1 = IND + 1
     IF (NSEC -IND-10) 963, 963, 964
963  WRITE OUTPUT TAPE 6, 103, IND, (T(I), I = IP1, NSEC)
     CALL EXIT
964  IP10 = IND + 10
     WRITE OUTPUT TAPE 6, 103, IND, (T(I), I = IP1, IP10 )
     IND = IND + 10
     GO TO 962
970  ZLAME = ZLMAX
     NUMB = NUMB + 1
     WRITE OUTPUT TAPE 6, 107, NUMB
     E = 0.0
     VAR = 0.0
     DO 971 I = 1, NP1
     X(I) = T(I) + U(I) * ZLAME
     VAR = VAR + Q(I) * X(I)**2
971  E = E + X(I) * A(I)
     STDEV = VAR ** .5
     Z2LAM = 2.0 * ZLAME
     WRITE OUTPUT TAPE 6, 105, E, STDEV, Z2LAM
     WRITE OUTPUT TAPE 6, 109
     IND = 0
972  IP1 = IND + 1
     IF (NSEC - IND - 10) 973, 973, 974
973  WRITE OUTPUT TAPE 6, 103, IND, (X(I), I = IP1, NSEC )
     GO TO 499
974  IP10 = IND + 10
     WRITE OUTPUT TAPE 6, 103, IND, (X(I), I = IP1, IP10)
     IND = IND + 10
     GO TO 972
     END
```

APPENDIX C

COMPANIES INCLUDED IN THE SAMPLE OF INDUSTRIAL COMMON STOCKS

This appendix lists the 96 companies in the sample of industrial common stocks described in Chapter III. A few of the companies had more than one issue of common stock outstanding during some of the years of the period studied; in such cases the series used in the study is identified under "Remarks", as are events which affected the names of the companies studied.

	Company	Remarks
1	Allegheny Ludlam Steel	
2	Allen Industries	
3	American Bosch	
4	American Car and Foundry, Inc.	1954: name changed to ACF Industries
5	American Encaustic Tiling	1958: merged with National Gypsum Co.
6	American Machine and Metals	
7	American Radiator and Standard Sanitary	
8	American Woolens	1958: merged into Textron, Inc.
9	Andes Copper Mining	prior to Feb. 1958: capital stock, par $14; after Feb. 1958; Class B stock, par $35
10	Armstrong Cork	
11	Barker Brothers (Md)	1958: merged into Barker Brothers (RI)
12	Blaw Knox	
13	Borden	
14	Borg Warner	
15	Brown Shoe Company, Inc.	
16	Canada Dry Ginger Ale	
17	Celanese Corporation of America	
18	Century Ribbon Mills	1956: name changed to Century Industries Company, Inc.
19	City Ice and Fuel	1949: name changed to City Products Corporation
20	Colgate Palmolive Company	
21	Commercial Solvents	
22	Conde Nast Publications	
23	Consolidated Voltee Aircraft	1954: merged with General Dynamics Corporation
24	Continental Can	
25	Continental Oil	
26	Corn Products Refining	1958: merged with Corn Products Company
27	Curtiss Wright	
28	Devoe and Reynolds Company	prior to Sept. 1959: $2 par Class A stock; after Sept. 1959; $2 par common stock
29	Dow Chemical	
30	Electric Autolite	

	Company	Remarks
31	Eureka Williams	1954: name changed to Wordell Corporation 1957: merged into Amerace Corporation
32	Firestone Tire and Rubber	
33	Flintkote Company (Mass.)	
34	Food Machinery and Chemical Corporation	
35	Gaylord Container	1955: merged into Crown Zellerbach Corporation
36	General Mills, Inc.	
37	General Tire and Rubber Co. (Ohio)	
38	Gotham Hosiery	1955: merged into Chadbourn-Gotham, Inc.
39	Guantanamo Sugar	
40	Hazel-Atlas Glass	1956: acquired by Continental Can Company, Inc.
41	Holland Furnace	
42	Inspiration Consolidated Copper Mines	
43	Interchemical Corporation	
44	Interstate Department Stores	
45	Johns Manville	
46	Julius Kayser	1958: name changed to Kayser-Roth Corporation
47	S. S. Kresge	
48	Kroger	
49	Lerner Stores	
50	Lorillard	
51	Mack Trucks	
52	Martin-Parry	1956: name changed to Ward Industries Corp.
53	Melville Shoe	
54	Mengel	
55	Midcontinent Petroleum	1955: merged into Sunray-Midcontinent Oil Company
56	Minneapolis Honeywell Regulator	
57	Motor Wheel	
58	National Biscuit	
59	National Malleable and Steel Castings	
60	Natomas	
61	New York Air Brake	
62	Oliver	

	Company	Remarks
63	Outboard Marine and Manufacturing	1956: name changed to Outboard Marine Corporation
64	Pacific Tin Consolidated Corporation	
65	Parke, Davis and Company	
66	Penick and Ford, Limited, Inc.	
67	Pennsylvania Coal and Coke	1954: name changed to Penn-Texas Corporation 1959: name changed to Fairbanks Whitney Corporation
68	Phelps Dodge Corporation	
69	Pittsburgh Coke and Iron	1944: name changed to Pittsburgh Coke and Chemical Company
70	Pressed Steel Car	1954: name changed to U. S. Industries, Incorporated
71	Reliance Manufacturing	
72	Remington Rand	1955: merged into Sperry-Rand Corporation
73	Ritter Company	
74	Savage Arms Corporation	
75	Sharon Steel Carporation	
76	Silver King Coalition Mines	1953: merged into United Park City Mines Company
77	Spear and Company	1960: name changed to Acme-Hamilton Manufactoring Corporation
78	Sun Oil	
79	Sweets Company of America, Inc.	
80	Texas Company	1959: name changed to Texaco, Inc.
81	Timken Detroit Axle	1953: merged into Rockwell Spring and Axle Company 1958: name changed to Rockwell-Standard Corporation
82	Twentieth Century Fox	
83	Twin Coach	
84	Union Bag and Paper	1956: name changed to Union Bag-Camp Paper Corporation
85	United Engineering and Foundry Company	
86	United States Freight	

	Company	Remarks
87	United States Rubber	
88	United States Tobacco	
89	United Stockyards	
90	Warren Foundry and Pipe	1956: name changed to Shahmoon Industries, Inc.
91	Waukesha Motor	
92	Western Auto Supply	
93	Wilcox Oil	
94	Wilson and Company	
95	L. A. Young Spring and Wire	1957: name changed to Young Spring and Wire
96	Zonite Products	1956: name changed to Chemway Corporation

APPENDIX D

SECURITIES PROVIDED FOR THE SUBJECTIVE PREDICTION EXPERIMENT

The appendix lists the 20 securities provided by an investment counselor as part of the experiment described in Chapter IV. The parameters of the diagonal model for each security are also shown, as are the two parameters which describe the distribution of the Dow-Jones Industrial Average. All values were computed in the manner described in Chapter IV from estimates provided by the investment counselor.

Number	Security	Parameters of the Diagonal Model: A_i	B_i	Q_i
1	American Machine and Foundry Common	1.539	2.334	0.297
2	American Telephone and Telegraph Common	1.424	1.279	0.088
3	Armour and Company Common	1.436	1.805	0.150
4	Beckman Instruments Common	1.553	2.613	0.349
5	Carolina Power and Light Common	1.290	1.196	0.064
6	Diamond National Common	1.511	1.473	0.143
7	El Paso Natural Gas Common	1.409	1.732	0.103
8	Florida Power and Light Common	1.297	1.820	0.120
9	General Precision Equipment $2.98 Preferred	1.553	2.469	0.202
10	Haloid Xerox Common	2.011	2.008	0.329
11	International Telephone and Telegraph Common	1.358	1.607	0.139
12	Kerr McGee Oil Industries Common	1.503	1.672	0.134
13	Northern Pacific Common	1.382	1.733	0.136
14	Peoples Gas, Light and Coke Common	1.312	1.098	0.055
15	Reynolds Metals Common	1.465	2.304	0.167
16	San Diego Gas and Electric Common	1.318	1.054	0.058
17	Southern California Edison Common	1.336	0.340	0.038
18	Toledo Edison Common	1.326	1.150	0.067
19	Union Electric Common	1.304	1.103	0.058
20	Universal Match Common	1.494	3.090	0.361
21	Dow-Jones Industrial Average	0.503		0.089