Technical Details and Further Intuition of Markowitz's (1952a) Paper, *Portfolio Selection*

By Andrew W. Lo and Stephen R. Foerster

www.InPursuitOfThePerfectPortfolio.com

July 8, 2021

Harry Markowitz's paper focused on the stage of portfolio selection that started with beliefs or expectations about future stock performance and then attempted to determine how one would choose a particular portfolio of risky stocks. He considered a rule whereby investors considered higher expected return (E) desirable and variance (V) or variability of return as undesirable, as indicated in our Figure 1.[1] In other words, investors would seek a portfolio with the highest amount of expected return for a given level of risk.

Much of the paper covered what Markowitz referred to as "elementary concepts and results of mathematical statistics." He first considered a random variable or random outcome, Y, whose value was determined by chance, such as the roll of a die. He defined the probability that $Y = y_1$ as p_1; that $Y = y_2$ as p_2; etc. He then defined the expected value of Y or its mean as:

$$E = p_1y_1 + p_2y_2 + \ldots + p_Ny_N.$$

The variance of Y was defined as:

$$V = p_1(y_1 - E)^2 + p_2(y_2 - E)^2 + \ldots + p_N(y_N - E)^2.$$

He noted that V, the average squared deviations from E was a commonly used dispersion measure. The only somewhat awkward stipulation was that variance was expressed in non-intuitive "percentage squared" terms. A related measure known as standard deviation or σ (the Greek letter sigma), which was the square root of variance or $\sigma = \sqrt{V}$, was expressed in the more familiar percentage terms.

At this point in his discussion, Markowitz was simply introducing general mathematical terms, but let us skip ahead for a moment, in order to further describe these principles in the context of holding

[1] We are using Markowitz's representation of graphing E on the horizontal axis and V on the vertical axis, whereas the more common practice today is the reverse.

one stock. We could describe stock A as having an expected return next year of 12 percent, with a standard deviation of 20 percent. For descriptive purposes only, if one were to assume a particular form of the distribution of returns—which it is important to note was *not* what Markowitz assumed—then one could make inferences about the return and risk tradeoffs for this particular stock. For example, if we assume the well-known Normal distribution or bell-shaped curve, then by the properties of that distribution, about two-thirds of the expected returns would occur plus or minus one standard deviation from the mean and about 95 percent of the expected returns would occur plus or minus two standard deviations from the mean. In this example, two-thirds of the time the investor would expect returns of between -8 percent (or 12 percent - 20 percent) and 32 percent (or 12 percent + 20 percent); and 95 percent of the time between -28 percent and 52 percent.

Back to generalities, Markowitz then considered the context of a number of random variables, R_1, R_2, …, R_n, such as rolling dice. He defined R, also a random variable, as simply a weighted sum (or linear combination) of the various individual R_is (with weights α_1, α_2, …, α_n):

$$R = \alpha_1 R_1 + \alpha_2 R_2 + \ldots + \alpha_n R_n.$$

For example, R_1 might be the number that turns up on the first die and R_2 the number that turns up on the second die, and R being the simple sum of these two numbers (with $\alpha_1 = \alpha_2 = 1$).

Markowitz noted that the expected value of a weighted sum is simply the weighted sum of the expected values. In other words:

$$E(R) = \alpha_1 E(R_1) + \alpha_2 E(R_2) + \ldots + \alpha_n E(R_n).$$

As Markowitz pointed out, "the variance of a weighted sum is not as simple." Markowitz introduced another important mathematical concept known as covariance. Covariance captured the extent to which the outcome of two random variables was related. The covariance of R_1 and R_2 was estimated as:

$$\sigma_{12} = E\{ [R_1 - E(R_1)] [R_2 - E(R_2)] \}.$$

Mathematically, the covariance was also expressed as:

$$\sigma_{12} = \rho_{12} \sigma_1 \sigma_2,$$

where ρ_{12} was the correlation between the returns of stock 1 and stock 2. Note that the correlation is a type of standardized covariance measure and is always between -1 (known as perfect negative correlation, whereby two stocks would always move in opposite directions) and +1 (known as perfect positive correlation, whereby two stocks would always move in synch).

With the covariance term defined, Markowitz showed that the variance of a random variable such as R depended not only on the variability (as captured by the variance or standard deviation) of each of the individual random variables R_1, R_2, etc., but also—and more importantly—on the covariances. In general terms, the variance of a random variable such as R that depended on N random variables was measured as:

$$V(R) = \sum_{i=1}^{N} \alpha_i^2 V(X_i) + 2 \sum_{i=1}^{N} \sum_{i>1}^{N} \alpha_i \alpha_j \sigma_{ij}.$$

Since another way to express the variance of R_i as σ_{ii}, then the equation above could be simplified as:

$$V(R) = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \sigma_{ij}.$$

Markowitz then considered the context of stocks with the random variable R_i now representing the return on stock i. The expected value or expected return for stock i was represented as μ_i. Recall that Markowitz did not deal with how one might estimate μ_i, which was what he referred to as the first stage of portfolio selection, however one approach might be to simply take some kind of historical average return for a particular stock and assume it is also one's best guess of the expected stock return. If the percentage of an investor's assets allocated to stock i was X_i, then the portfolio return, R, would be a simple weighted average or linear combination of the random variables:

$$R = \sum_{i=1}^{N} R_i X_i.$$

Markowitz noted that the weights themselves, the various X_is, were not random variables but rather were fixed by the investor. He also noted since these weights were percentages, they all had to add to 1. He excluded the possibility of any short sales and as such assumed that all weights were non-negative (i.e., equal to or greater than zero). Since the return on the portfolio as a whole, R, was a weighted sum of random variables, and hence R was itself a random variable, then by the mathematical properties described above, the expected return, E, for the entire portfolio was:

$$R = \sum_{i=1}^{N} X_i \mu_i,$$

and the variance was:[2]

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} X_i X_j.$$

We present an example here to illustrate a simple two-stock portfolio. Suppose stock 1 has an expected return of 8 percent, stock 2 has an expected return of 14 percent, and the investor chooses to invest 60 percent of her assets in stock 1 and 40 percent in stock 2. The resulting expected portfolio return is (0.6)(8 percent) + (0.4)(14 percent) or 10.4 percent. Suppose also that the investor believes that the standard deviation of returns for stock 1 is 20 percent (and consequently the variance is the squared value of this amount), the standard deviation of returns for stock 2 is 30 percent, and the correlation between the returns of stock 1 and stock 2 is 0.25. Note that a simple weighted-average of the two standard deviations is (0.6)(20 percent) + (0.4)(30 percent) = 24.0 percent. Markowitz's main insight was that the actual resulting portfolio standard deviation should be *less* than this simple weighted-average, so long as the correlation between the two stocks is less than 1.0 (in the special case of perfect positive correlation, then the resulting portfolio standard deviation would be the simple weighted-average of 24 percent). Using the formulas above, the covariance between stocks 1 and 2 is (0.25)(0.20)(0.30) or 0.015, and the portfolio variance is $(0.20)^2(0.6)^2 + (0.30)^2(0.4)^2 + 2(0.015)(0.6)(0.4) = 0.036$. The resulting portfolio standard deviation is the square root of the variance or 19.0 percent—as expected, less than the simple weighted-average of 24.0 percent.

Given the fixed probability beliefs of the investor, i.e., the set of μ_is and σ_{ij}s or the set of expected returns, variances (or standard deviations), and covariances, the investor had a choice of various E and V combinations depending on his choice of $X_1, X_2, \ldots, X_N$. As Markowitz showed in our Figure 2, there is an attainable set of E-V combinations—shown in the oval—but also an efficient sub-set that have minimum variance for a given E or maximum E for a given V. Thus, Markowitz showed that the portfolio selection problem has been greatly simplified through such an approach.

[2] In the original paper, there appears to be a small typo with the "j" subscript in the equation missing.

Markowitz indicated that there were techniques available by which to calculate the efficient set, but did not present the techniques in the paper. Instead, he used geometry to illustrate the case of a three-stock portfolio.

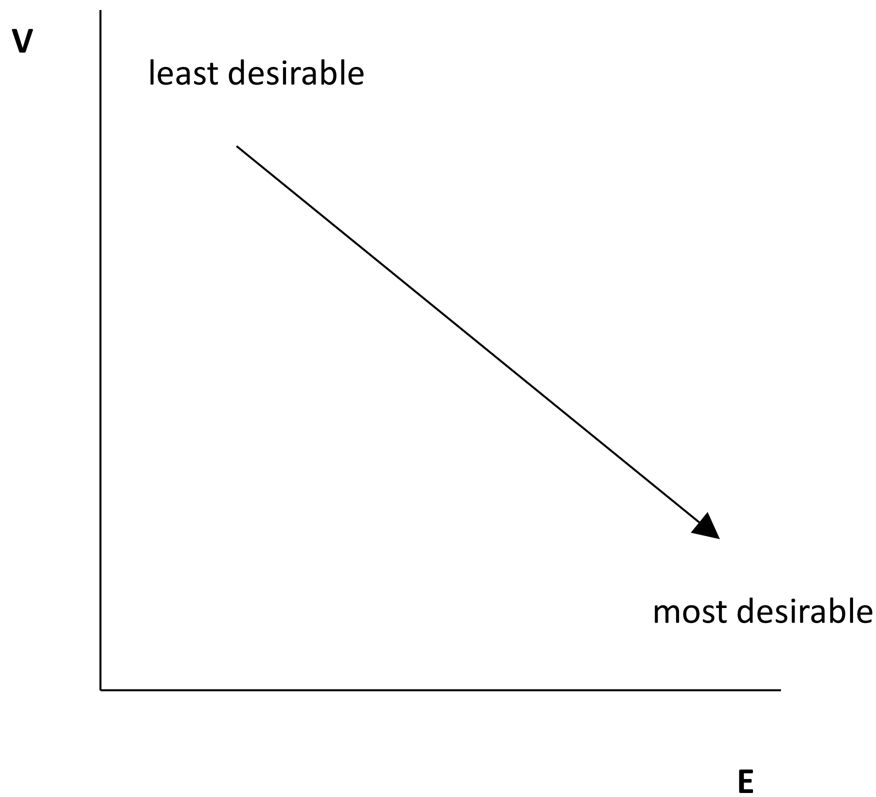

Figure 1

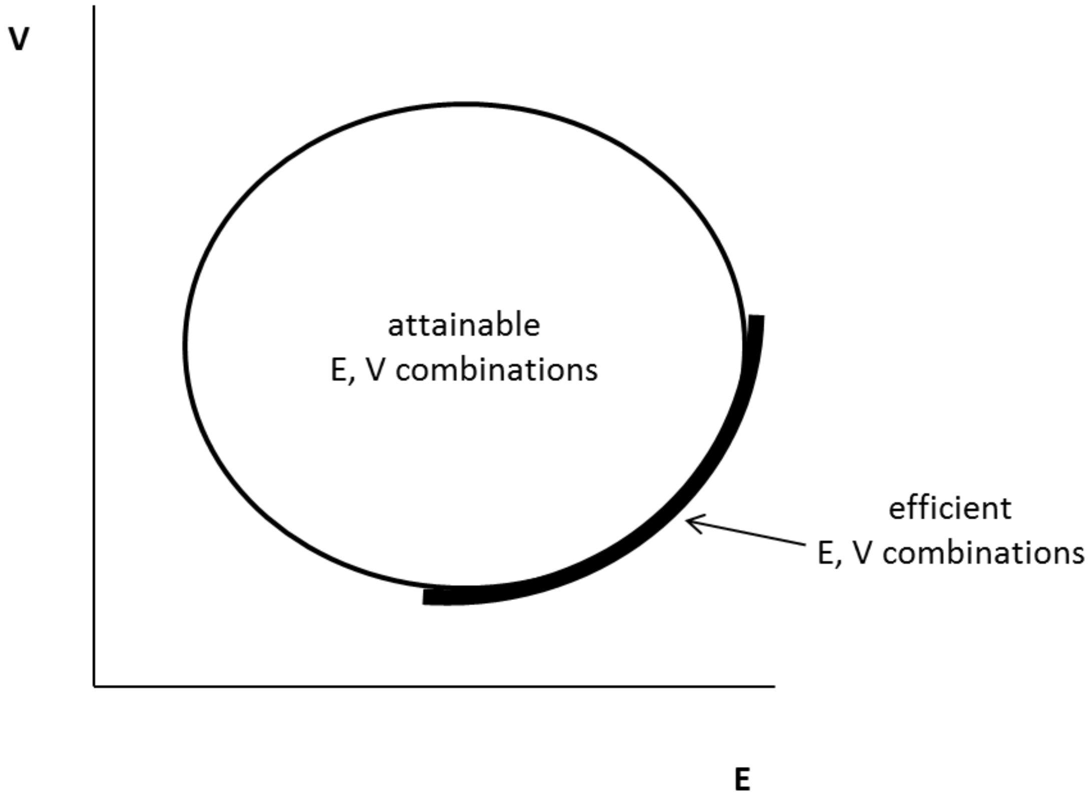

Figure 2